Dynamic Fractals

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Dynamic Fractals

1. *By hand* (no cheating!), use the Babylonian method $x_n = \frac{1}{2} \left( x_{n-1} + \frac{3}{x_{n-1}} \right)$ to find the first few terms of a rational sequence that approaches $\sqrt{3}$ as its limit. Try to quantify the rate at which your sequence approaches its limit.

2. If $p$ and $q$ are positive numbers, then their geometric mean is $\sqrt{pq}$ and their arithmetic mean is $\frac{1}{2}(p + q)$. Prove that the arithmetic mean is never less than the geometric mean. When do the two agree? What does this have to do with your root-finding investigation?

3. Suppose that $x$ is positive, and that $y$ is between $x$ and $3/x$. Demonstrate that $3/y$ must also be between $x$ and $3/x$. What bearing does this have on our root-finding investigation?

4. Prove that $\sqrt{3}$ is between $x$ and $3/x$, for any positive number $x$. What bearing does this have on your root-finding investigation?

5. Given a function $f$, such as

$$f(x) = \frac{1}{2} \left( x + \frac{3}{x} \right),$$

and a value $x_0$, you may recursively generate the sequence $x_0, x_1, x_2, \ldots$, by calculating $x_n = f(x_{n-1})$ for all positive integers $n$. The value $x_0$ is called a seed value. The subject of dynamic systems is concerned with the limiting values of sequences built in this way. In particular, it will be interesting to see how the choice of the seed $x_0$ affects the outcome. Write a short program that will produce numerical data of the sort you are investigating.

6. What would happen if you were to try the Babylonian method on the problem of finding a square root of $-1$? Try it out anyway! What sort of a sequence do you get? Can you make sense of it? Try the seed value $x_0 = 2$. Try the seed value $x_0 = 1/\sqrt{3}$.

7. It is also natural to try to extend the Babylonian approach to the problem of finding cube roots. Design a dynamic system that finds the cube root of 3. Test your method. If it does not work, fix it. If it does not work well, make it work better.

8. Simplify $\cot \theta - \tan \theta \over 2$, and relate this expression to a root-finding investigation.

9. The earliest versions of hand-held calculators (*circa* 1972) had only the basic four arithmetic operations, although some were equipped with a square-root key. If you had such a square-root calculator, you could actually use it to evaluate more complicated functions. For example, explore the sequence defined by $x_n = \sqrt{3x_{n-1}}$, taking $x_0$ to be a positive seed value. What is the limit of this sequence, and does it depend on $x_0$?
Dynamic Fractals

Much of your work will take place in the familiar two-dimensional coordinate plane. It will often be convenient, however, to look at the plane as the one-dimensional domain of complex numbers. The standard approach is to identify the point \((x, y)\) with the complex number \(x + yi\). To emphasize the one-dimensionality of this algebra, it is customary to write \(z\) as an abbreviation for \(x + yi\). The real part of \(z\) is \(x\), and the imaginary part is \(y\). Notice that the imaginary part is a real number! The familiar notation \(|z|\) denotes the absolute value of the complex number \(z\), also known as the magnitude of \(z\). In other words, \(|z|\) is the length of \(z\) (considered as a vector), or the distance from \(z\) to the origin.

It is also possible to describe complex numbers in terms of polar coordinates, by writing \(z = r(\cos \theta + i \sin \theta) = r \text{cis} \theta\). The polar variable \(r\) is just the magnitude \(|z|\).

1. Show that each of the following can be rewritten in \(a + bi\) form:
   (a) \((3 + 4i) + (2 + i)\)  
   (b) \((3 + 4i) - (2 + i)\)  
   (c) \((3 + 4i) \cdot (2 + i)\)  
   (d) \((3 + 4i)/(2 + i)\)

2. The conjugate of a complex number \(z = a + bi\) is the number \(\overline{z} = a - bi\), provided that \(a\) and \(b\) are real. Show that \(z\overline{z}\) and \(z + \overline{z}\) are always real numbers.

3. Explain why it is not correct to think of complex as a synonym for non-real.

4. Convert each of \(1 + i\), \(2 + i\), and \((1 + i)(2 + i)\) to polar form. Notice anything?

5. Describe the solutions to the following, as configurations in the plane:
   (a) \(|z| = 3\)  
   (b) \(|z - 2| = 3\)  
   (c) \(|z + 1| = |z - 2|\)  
   (d) \(|z - 2| = |z - i|\)  
   (e) \(i|z|\) is real  
   (f) \((2 + i)z\) is real  
   (g) \(z^2\) is real  
   (h) \(|z^2| = |z|^2\)

6. Considering \(x\) and \(y\) to be real, find the real and imaginary parts of the following:
   (a) \((x + yi)^2\)  
   (b) \(\frac{3 + 4i}{x + yi}\)  
   (c) \((1 + i)^{25}\)  
   (d) \(i^{1994}\)

7. In calculus courses, unit complex numbers \(\cos \theta + i \sin \theta\) are usually written as \(e^{i\theta}\). Use your knowledge of infinite series to justify this so-called Euler identity.

8. Find two ways of justifying the identity \(\text{cis}(\alpha + \beta) = \text{cis}(\alpha)\text{cis}(\beta)\).

9. Any complex number can be expressed in the form \(re^{i\theta}\). This implies that any nonzero complex number can be expressed in the form \(e^{a+bi}\). For example, \(-1 = e^{i\pi}\). It follows that nonzero numbers all have logarithms — infinitely many logarithms, in fact.
   (a) What are the logarithms of \(-1\)?
   (b) Find the logarithms of \(i\), then use them to show that \(i^i\) has infinitely many different values, all of which are real.
Dynamic Fractals

1. A small but interesting fact from algebra: If $a$, $b$, $c$, and $d$ denote positive quantities, then $(a + c)/(b + d)$ lies between $a/b$ and $c/d$. Verify.

2. The preceding interpolation result provides another way of designing a square-root-finding process. As before, you seek a number that is guaranteed to be between $x/1$ and $3/x$; this time, take $(x + 3)/(1 + x)$. In other words, it seems that iterating the function $f(x) = \frac{x + 3}{x + 1}$ ought to produce sequences that approach $\sqrt{3}$ as a limit. Investigate the performance of this algorithm.

3. Suppose that a recursively defined sequence $x_n = f(x_{n-1})$ converges to $p$. For most functions $f$, it is guaranteed that $p$ be a fixed point of $f$ — in other words, that $f(p) = p$. What property of $f$ is responsible for this guarantee?

4. Draw the graphs of $y = x$ and $y = \cos x$ on the same coordinate-axis system. It should be clear that there is a single point of intersection, which can not be found by traditional methods of equation-solving. Nevertheless, it is easy to find accurate coordinates for the intersection point, by using a dynamic approach to calculation. Try it.

5. If $x$ is a positive quantity, and if $y$ is between $x$ and $3/x^2$, need it be true that $3/y^2$ is also between $x$ and $3/x^2$?

6. Prove that the dynamic system defined by $f(x) = \frac{1}{3} \left( 2x + \frac{3}{x^2} \right)$ produces sequences that converge to the real cube root of 3, for all positive seed values, and most negative seed values, too. Do you see where this function came from? What are its fixed points?

7. Prove that the Babylonian square-root process does indeed converge quadratically. Hint: Write the function in the form $f(x) = \frac{1}{2} \left( x + \frac{m^2}{x} \right)$, where $m$ stands for the target root, then compare two successive errors, $x - m$ and $f(x) - m$.

8. It seems plausible that the search for a square root of $-1$ would succeed, if a non-real seed value were provided. Investigate this. It will probably help to write a little program to produce the numerical data, and this will require that you keep track of the real and imaginary parts separately. Because non-real values are allowed, it is also appropriate to change letters and describe the root-finding function as $f(z) = \frac{1}{2} \left( z - \frac{1}{z} \right)$.

9. Investigate the iterative properties of $f(x) = \frac{x^3 + 9x}{3x^2 + 3}$.

10. An important calculus question: If $f(2) = 7$ and $f'(2) = -0.5$, then what can be said about $f(2.3)$?
Dynamic Fractals

1. Given a recursively defined sequence \( x_n = f(x_{n-1}) \), convergence can depend on the choice of seed value \( x_0 \). Illustrate.

2. Suppose that \( f \) is a linear function, with \( f'(2) = 0.4 \) and \( f(2) = 2 \). Investigate the rate of convergence of the sequence defined recursively by \( x_0 = 3 \) and \( x_n = f(x_{n-1}) \). Do the same for the sequence initiated by \( x_0 = -5 \).

3. Verify that \( \text{cis}(144) \) is one of the solutions to \( z^5 = 1 \). What are the others?

4. The simplest example of quadratic convergence is provided by — you guessed it — the function (i.e., dynamic system) \( f(x) = x^2 \). Explain.

5. What if the preceding example had been defined as \( f(z) = z^2 \)?

6. Write a definition of what it means for the convergence of a sequence to be cubic.

7. Thanks to the Euler identity, the expression \( e^z = e^{x+yi} \) is defined for any complex number \( z \). Show that this function is periodic, by finding its period.

8. Find a way of showing that \( \cos yi = \cosh y \) and \( \sin yi = i \sinh y \). These identities should help explain why the hyperbolic functions and the circular functions have virtually the same formal properties.

9. Extending the definitions of the circular and hyperbolic functions to allow complex values is now a routine matter. Verify that the real and imaginary parts of these functions are as follows:

\[
\begin{align*}
\cos(x + yi) &= \cos x \cosh y - i \sin x \sinh y \\
\sin(x + yi) &= \sin x \cosh y + i \cos x \sinh y \\
\cosh(x + yi) &= \cosh x \cos y + i \sinh x \sin y \\
\sinh(x + yi) &= \sinh x \cos y + i \cosh x \sin y 
\end{align*}
\]

10. The limit of the sequence \( x_n = \left(\frac{1}{17}\right)^n \) is zero. Describe the rate of convergence.

11. The limit of the sequence \( x_n = \left(\frac{1}{7}\right)^{2^n} \) is zero. Describe the rate of convergence.

12. Give an example of an explicitly defined sequence that approaches its limit at a cubic rate.

13. An algebra student was asked to solve \( x^2 + 3x - 5 = 0 \) for \( x \), and responded by writing \( x = (5 - x^2)/3 \). Explain why this answer is actually more useful than it may have looked to the student’s algebra teacher. Are there other useful (but unconventional) responses to the question?
Dynamic Fractals

1. Suppose that $f(2) = 7$ and $f'(2) = \frac{1}{2}i$. What can be said about $f(2.3)$?

2. The complex numbers $z_1 = 1 + 2i$, $z_2 = 5 + 5i$, and $z_3 = 11 - 3i$ mark the corners of a right triangle. What about the three numbers $\frac{1}{2}iz_1$, $\frac{1}{2}iz_2$, and $\frac{1}{2}iz_3$?

3. Given a configuration of points (finite or infinite) in the complex plane, what is the effect of multiplying every point by $\frac{1}{2}i$?

4. Suppose that a sequence $x_n$ approaches its limit $L$ in such a way that $|x_n - L|$ is one tenth of $|x_{n-1} - L|$, for all positive $n$. Describe the rate of convergence of this sequence.

5. Write a function that will recursively generate sequences that approach the square roots of $3 + 4i$ (there are two). Try your method on a variety of seed values. Programming your calculator to crunch the numbers is advisable; see item 8 on page 206. This investigation has many goals. One of the goals is to find the square roots, of course. Another goal is to observe the rate of convergence. Another goal is to discover whether there are any seed values for which your algorithm does not converge. Another goal is to find a way of predicting what will happen once a particular seed value is chosen.

6. Both functions

$$f(x) = \frac{1}{2} \left( x + \frac{5}{x^2} \right) \quad \text{and} \quad g(x) = \frac{1}{3} \left( 2x + \frac{5}{x^2} \right)$$

can be used recursively to find the cube root of 5. The latter function produces sequences that converge quadratically to the root, while the former produces sequences that converge only linearly to the root. In either case, the ratios

$$\frac{x_n - \sqrt[3]{5}}{x_{n-1} - \sqrt[3]{5}}$$

approach limits as $n$ approaches infinity. Make calculations that confirm this.

7. With $f$ as defined in the preceding item, graph both of the equations $y = x$ and $y = f(x)$ on the same coordinate axes (which are not in the complex plane). What is the significance of the point where these graphs intersect? What is the significance of the limit found in the preceding item?

8. Repeat the preceding graphical analysis on the slow square-root finder $f(x) = \frac{x + 3}{x + 1}$, introduced in item 2 on page 3. In particular, notice the significance of the two intersections of the graph $y = x$ and the graph $y = f(x)$, and find the source of the 27% figure observed on page 204.
Dynamic Fractals

1. The Babylonian method for finding square roots has evolved into a very refined form, known as the Newton-Raphson method for solving equations. In principle, it applies to any equation \( E(x) = 0 \) that is defined by a differentiable function \( E \). As the figure suggests, the method tries to improve a given approximation \( x = a \) by using the tangent line at \((a, E(a))\) to calculate a new approximation \( x = b \). Show that

\[
b = a - \frac{E(a)}{E'(a)}.
\]

2. It is evident that the Newton-Raphson method — like its precursor, the Babylonian method — can be applied again and again to any seed value. In other words, given an equation \( E(x) = 0 \), the function

\[
N(x) = x - \frac{E(x)}{E'(x)}
\]

defines a dynamic system, whose fixed points are precisely the solutions to the equation. In effect, this dynamic approach is to let the equation solve itself, there being no special formulas or tricks that are used to find the roots.

Set up the dynamic system function \( N \) for each of the following equations:

(a) \( x^2 = 3 \)  
(b) \( x^3 = 5 \)  
(c) \( x^n = A \)  
(d) \( x = \cos x \)  
(e) \( x^2 + 3x = 5 \)

3. You will often have occasion to write iterated function evaluations, such as

\[
\cos(\cos(\cos(\cos(1)))))) \quad \text{and} \quad f(f(f(f(f(x_0)))))).
\]

Suggest a notation that makes it easier to read and write such expressions.

4. Suppose that \( f(\sqrt{3}) = \sqrt{3} \), \( f'(\sqrt{3}) = 1.6 \), and that a sequence is defined by \( x_0 = 1.732 \) and \( x_n = f(x_{n-1}) \) for all positive \( n \). What can you say about the value of \( x_{21} \)?

5. If \( N \) is the Newton-Raphson function associated with an equation \( E(x) = 0 \) (see item 2 above), and if \( m \) is any simple (non-repeated) solution to this equation, then \( N'(m) = 0 \). Prove this remarkable property of the function \( N \). What implications does this result have for dynamic equation solving?

6. Apply the Newton-Raphson method to the equation \( x^3 - 3x^2 + 3x - 1 = 0 \). Is there anything remarkable about this example?
Dynamic Fractals

To graph a real-valued function of a real variable, two real dimensions are needed — one for the domain and one for the range. To graph a complex-valued function of a complex variable, four real dimensions are needed — two for the domain and two for the range. It is therefore clear that graphical analysis of complex functions must proceed in a different manner, and that to visualize such a function is a challenging endeavor. Figure 1 presents one way to picture the root-finding function

\[ f(z) = \frac{1}{2} \left( z + \frac{3 + 4i}{z} \right), \]

which was introduced in item 5 on page 5.

The origin lies at the center of the window, and the roots \(2 + i\) and \(-2 - i\) are near the centers of the two small white circles. The figure is divided by the line \(y = -2x\), which is the perpendicular bisector of the segment joining the two roots. The shaded bands (which are in fact bounded by circles) help to display the function \(f\) as a dynamic object, for they record the convergence of sequences \(\{z_n\}\) towards their limits — each application of \(f\) moves the all the points of a band into a neighboring band. There are infinitely many such bands that surround each of \(2 + i\) and \(-2 - i\). The radius of the two smallest circles is approximately 0.1. The large circles bunch together so tightly near the line \(y = -2x\) that they are indistinguishable from the line and from each other.

1. What is the significance of the crowded arrangement of circles near the line \(y = -2x\)?

2. All the bands in the figure are determined by the radii of the tiny circles that surround the fixed points. What would be the effect on the figure of changing these radii to a smaller value?

3. You have seen evidence that \(z\)-values that lie on the line \(y = -2x\) initiate sequences that are confined to this line. \((z = 2 - 4i,\) for example — see page 211). Prove analytically that \(f(z)\) is equidistant from \(2 + i\) and \(-2 - i\) whenever \(z\) is. Hint: It is easier to work in a general setting, by letting

\[ f(z) = \frac{1}{2} \left( z + \frac{m^2}{z} \right); \]

prove that \(f(z)\) is equidistant from \(m\) and \(-m\) whenever \(z\) is.

4. Find both values of \(z\) that satisfy the equation \(f(z) = \frac{1}{4}(2 + i)\).
Dynamic Fractals

Let $A = (3, 0)$ and $B = (-3, 0)$ for the first three questions.

1. Plot all the points $P = (x, y)$ that are twice as far from $A$ as they are from $B$. In other words, plot solutions to the equation $PA = 2 \cdot PB$. In particular, identify the points $P$ that are on the $x$-axis. By using the distance formula, show that this configuration is a circle, and find its center and its radius.

2. Plot the solutions to $PA = \frac{1}{2} \cdot PB$. This is another circle, of course.

3. Given a positive real number $k$, the equation $PA = k \cdot PB$ defines one of the circles of Apollonius. In terms of $k$, write formulas for the center and the radius of the circle. Notice that $k = 1$ defines a line, so that your formulas should not make sense when $k = 1$.

4. Let $m$ stand for an arbitrary nonzero complex number ($m = 2 + i$, for example), and $k$ be a positive real number. Consider all those $z$ that fit the equation $|z - m| = k \cdot |z + m|$. Make use of what you have learned from the preceding three questions to write formulas for the center and the radius of this circle. It is not necessary to do any more calculations.

Given almost any seed value $z_0$, the square-root-finding function $f(z) = \frac{1}{2} \left( z + \frac{m^2}{z} \right)$ generates a sequence $\{z_n\}$ that converges either to $m$ or to $-m$, The time has come to prove this statement!

5. Write simplified expressions for $|f(z) - m|$ and $|f(z) + m|$.

6. Suppose that $z$ fits the equation $|z - m| = k \cdot |z + m|$, for some positive number $k$. The preceding item implies that $|f(z) - m| = j \cdot |f(z) + m|$, for a certain number $j$ that depends only on $k$. Express $j$ in terms of $k$. Describe this situation in terms of Apollonian circles.

7. Use your findings to finish the convergence proof. Is it accurate to use the word monotonie to describe the convergence?

8. Explain why $z - m = z + m$ and $|z - m| = |z + m|$ say entirely different things about $z$. 
Dynamic Fractals

1. Use the quadratic formula to find both solutions to \(2z^2 + (2 - 3i)z = 1 + 2i\).

2. It is useful to think of complex numbers as vectors; in other words, the real and imaginary components of \(x + yi\) are \(x\) and \(y\), respectively. What is the size of the angle formed by \(2 + i\) and \((2 + i)(1 + i) = 1 + 3i\)?

3. Given the square-root finder \(f(z) = \frac{1}{2} \left(z + \frac{3 + 4i}{z}\right)\), verify that
   (a) \(f(1) = f(3 + 4i)\);
   (b) \(f(1 - i) = f(-0.5 + 3.5i)\);
   (c) there are two solutions to \(f(z) = 2 + 2i\);
   (d) for almost all \(w\), there are two solutions to \(f(z) = w\).

4. Isolate the real and imaginary parts of the function \(f(z) = \frac{1}{3} \left(2z + \frac{8}{z^2}\right)\), given that \(z = x + yi\). Can you guess what these will be used for?

5. Let \(f\) be a differentiable dynamic system. A fixed point \(p\) is attractive if \(|f'(p)| < 1\), repelling if \(1 < |f'(p)|\), superattracting if \(f'(p) = 0\), and indifferent if \(|f'(p)| = 1\). Explain this terminology by giving examples.

6. Given the square-root finder \(f(z) = \frac{1}{2} \left(z + \frac{3 + 4i}{z}\right)\), find the two \(z\)-values that make \(f(z) = 1 - 2i\). What is the significance of these numbers?

7. Figure 2 displays some of the dynamic behavior of the system defined by

\[
f(z) = \frac{z + 3 + 4i}{z + 1}.
\]

The origin is at the center of the window, and the bull’s-eyes are centered at \(2 + i\) and \(-2 - i\). What do the shaded bands mean for this example?

8. Let \(f\) be as in the preceding item. Even though they do not stand out in Figure 2, there are infinitely many exceptional seed values \(z_0\) that could present computational difficulty during the drawing process. Find three of them.

9. In the \(z^2 = 3 + 4i\) example illustrated in Figure 1, the line \(y = -2x\) has a special property: It is an example of an invariant set of points, meaning that the dynamic process carries each point in this set to another point in this set. Although the line \(y = -2x\) is a conspicuous invariant set, it is not the only invariant set. Find others.
Dynamic Fractals

1. Consider again the Newton-Raphson function \( f(z) = \frac{1}{2} \left( z + \frac{3+4i}{z} \right) \).

(a) The line \( y = -2x \) is an invariant set for the root-finding process. The sequence defined by the seed value \( z_0 = 2 - 4i \) stays on the line, therefore. (The computer data on pages 211 and 215 shows that this is a theoretical result.) Does the list \( z_0, z_1, z_2, \ldots \) fill the line, or are there points on the line that do not appear? Hint: \( z_2 = -\frac{7}{24} + \frac{7}{12}i \)

(b) In item 3 on page 9, you found that almost every complex number \( w \) has two ancestors, which are solutions to the equation \( f(z) = w \). These two \( z \)-values are also called inverse images of \( w \), and could be collectively denoted \( f^{-1}(w) \). In particular, 0 has two ancestors, \( 1 - 2i \) and \( -1 + 2i \). Along with 0, these lie on the line \( y = -2x \). Each of these ancestors of 0 also has two ancestors, which happen to lie on the same line (see item 6 on page 219). This enumeration of ancestors could continue forever. Explain why all the points found must lie on the line \( y = -2x \).

2. The function \( f(x) = \sin x \) provides a simple example of an indifferent fixed point. Show that, no matter what seed value \( x_0 \) is chosen, the sequence defined recursively by \( x_n = \sin x_{n-1} \) will approach 0 — very slowly. Use the graphical approach (known as a web diagram) illustrated on pages 212 and 213. Is the rate of convergence linear?

3. The quadratic function \( f(x) = x^2 + \frac{1}{4} \) provides another simple example of an indifferent fixed point. What is the fixed point? Describe the behavior of the sequences seeded by \( x_0 = 0 \) and by \( x_0 = 0.5000000001 \).

4. Given a nonzero complex number \( m \), what does the inequality \( |z - m| < |z + m| \) define? What about \( |z + m| < |z - m| \)?

5. The equation \( z^3 = 8 \) has three roots, \( 2, -1 + i\sqrt{3}, \) and \( -1 - i\sqrt{3} \), which can be found by a variety of means. They are fixed points for the Newton-Raphson root-finder

\[
f(z) = \frac{1}{3} \left( 2z + \frac{8}{z^2} \right),
\]

whose real and imaginary parts appear in item 4 on page 218. Using three colored pencils, and your calculator to gather data, produce a color map for this dynamic system. In other words, assign a different color to each of the three target roots — which are superattractors for the root-finding process — and color other seed values \( z_0 \) according to which target the sequence \( \{z_n\} \) approaches. For most seed values, it will not take many steps to make the color decision. There are many seed values that do not get a color because they do not lead to a root, but — except for one — they are irrational points, and therefore impossible to encounter when calculating with a machine.

Except for color, your finished map should display threefold symmetry.
Dynamic Fractals

Consider again the Newton-Raphson function \( f(z) = \frac{1}{2} \left( z + \frac{m^2}{z} \right) \). It is remarkably difficult to prove that \( \{z_n\} \) converges to \( m \) whenever the seed value \( z_0 \) is closer to \( m \) than it is to \(-m\). The difficulty is that the convergence you seek is not uniform — the closer \( z_0 \) is to the line \( |z - m| = |z + m| \), the more erratic are the first few terms of the sequence, making it impossible to write one analysis that covers all cases. Recall the sequence \( \{z_n\} \) on page 211, for example — the initial terms are very close to the line \( |z - m| = |z + m| \), which makes them behave in a bewildering manner. Here are some explorations that can be done using simple algebra and geometry:

1. Although this complex dynamic system is a straightforward extension of the Babylonian method, it is no longer possible to appeal to the trapping theme (see page 1) that explained the success of the real-variable algorithm. In other words, it is incorrect to think of \( m \) as being trapped between \( z \) and \( \frac{m^2}{z} - 1 \). Illustrate this by considering the example \( m = 2 + i \) and \( z_0 = 1 \).

2. Prove that the circle interior \( |z - m| < \frac{1}{2}|m| \) is an invariant set.

3. If \( |z_0 - m| < \frac{1}{2}|m| \), then the errors \( |z_n - m| \) decrease steadily to zero. In fact,
   \[
   |z_n - m| < \frac{|m|}{2} \implies |z_{n+1} - m| < \frac{|z_n - m|}{2},
   \]
   so that the errors decrease to zero at least as fast as \((\frac{1}{2})^n|z_0 - m|\). Prove this.

4. Prove that the circle interior \( |z - m| < \frac{2}{3}|m| \) is an invariant set.

5. The proposition in item 3 can be improved slightly, by enlarging the circle that encloses favorable seed values: If \( |z_0 - m| < \frac{2}{3}|m| \), then the sequence of errors \( |z_n - m| \) decreases steadily to zero.

Thus, if \( z_0 \) is within a suitably small circle centered at \( m \), then the resulting sequence \( \{z_n\} \) converges to \( m \) in a steady fashion, meaning that the error diminishes monotonically to zero. A slight modification to item 7 on page 5 (changing \( x \) to \( z \)) shows that the convergence is actually quadratic, of course. Despite what Figure 1 suggests, what happens to sequences (most sequences, that is) that are seeded from outside this suitably small circle is not yet clear.

You will soon find a revealing way to prove that \( \{z_n\} \) converges to whichever root is closer to \( z_0 \), using a method that does not depend on monotonicity.

6. The seed value \( z_0 = \frac{1}{3}(1 - 2i)\sqrt{3} \) provides an example of the strange things that occur inside the Julia set \( y = -2x \). Have you seen this example before, however?
Dynamic Fractals

You have been concentrating on what happens to a seed point when an algorithm is applied to it repeatedly. You can also look backwards, at what might be called the ancestors of a given point. In general, a point does not have a unique ancestor, so you must think in terms of an ancestral tree of points. For instance, the ancestral tree of the origin is of interest when you consider the Newton-Raphson examples, for these are the points that will eventually be sent to infinity, and therefore they will not converge to a root. There are a lot of them, and they are all in the Julia set. You have investigated the ancestral tree of 0 for the square-root finder, in item 1 on page 10 and item 6 on page 9 — now consider cube roots:

1. Consider \( N(z) = \frac{1}{3} \left( 2z + \frac{8}{z^2} \right) \), the Newton-Raphson cube-root finder. There are three immediate ancestors of 0, namely \( \sqrt[3]{4} \text{cis}(60) \), \( \sqrt[3]{4} \text{cis}(180) \), and \( \sqrt[3]{4} \text{cis}(300) \). Verify this, then consider the problem of locating the previous generation. It is not difficult to find them in Figure 3, but computing them requires a lot of work.

2. Verify that the function \( N \) in the preceding is three-to-one. What are the exceptional points, and in what way are they special?

Although the diagram for the cube-root finder is infinitely more complicated than the square-root diagram, it does a remarkable job of displaying the ancestral tree of zero. Indeed, it is precisely the ancestors of zero that you see when you look at this diagram. This does not happen in the square-root diagram, in which the Julia set is straight and featureless — every Julia point looks like every other, despite their different behaviors.

3. Use your calculator to verify that \( z = 1.07721735 \pm 0.83440897i \) is a 2-cycle for the cube-root finder \( N \). These are decimal approximations for \( z = \frac{2}{\sqrt[3]{10}} \text{cis} \left( \pm 60 \pm \frac{1}{3} \arctan \sqrt{\frac{27}{5}} \right) \).

4. Another detail from Figure 3: Each of the three basins of attraction is made up from infinitely many components of various sizes. The three largest (unbounded) ones contain the attractors and are invariant. The others are not invariant and are transformed onto one another (preserving colors, of course) when \( N \) acts. In particular, three of the components (one of each color) are transformed onto the invariant components; find them in the figure. In each of these components, one of the points must be an ancestor of its attractor! Find a precise description of these ancestors, in terms of the attractors (which — as usual — you can indiscriminately call \( m \)). Knowing this simple formula allows you to look at the diagram and quickly estimate its scale.

5. Ancestors of the three cube roots (the immediate ones were requested in the preceding item) are examples of points that are called eventually fixed. Explain this terminology. Did you encounter this phenomenon when you looked at square roots?
Dynamic Fractals

The Cubic Formula.

Although it is too complicated to be consistently of practical value, the cubic formula does provide theoretically correct solutions, which are occasionally of interest.

1. The first step in applying the formula is to bring the equation to the form \( z^3 + az + b = 0 \). This is a routine matter of shifting the variable right or left until the quadratic term disappears — completing the cube, you might call it.

2. Next, calculate the discriminant \( d = \frac{a^3}{27} + \frac{b^2}{4} \).

3. Set \( p = \sqrt[3]{\frac{-b}{2} + \sqrt{d}} \) and \( q = \sqrt[3]{\frac{-b}{2} - \sqrt{d}} \). If \( p \) and \( q \) are non-real (in other words, if \( d < 0 \)), then \( p \) and \( q \) should be conjugates.

4. The solutions to the equation \( z^3 + az + b = 0 \) are

\[
\begin{align*}
z &= p + q, \\
-\frac{p+q}{2} + \frac{p-q}{2}i\sqrt{3}, \\
-\frac{p+q}{2} - \frac{p-q}{2}i\sqrt{3}.
\end{align*}
\]

Examples.

5. To solve \( w^3 - 3w^2 + 9w - 9 = 0 \), start by replacing \( w \) by \( z + 1 \) — this produces the equation \( z^3 + 6z - 2 = 0 \). The discriminant is \( d = 8 + 1 = 9 \), so \( p = \sqrt[3]{4} \) and \( q = -\sqrt[3]{2} \). The solutions are

\[
\begin{align*}
z &= \sqrt[3]{4} - \sqrt[3]{2}, \\
z &= \frac{1}{2} \left( \sqrt[3]{2} - \sqrt[3]{4} \right) + \frac{1}{2} \left( \sqrt[3]{4} + \sqrt[3]{2} \right) i\sqrt{3}, \text{ and} \\
z &= \frac{1}{2} \left( \sqrt[3]{2} - \sqrt[3]{4} \right) - \frac{1}{2} \left( \sqrt[3]{4} + \sqrt[3]{2} \right) i\sqrt{3}.
\end{align*}
\]

The solutions to the \( w \)-equation are obtained by simply adding 1 to these \( z \)-values.

6. In item 1 on page 12, the three ancestors of \( -\sqrt[3]{4} \) were found by solving the equation \( 2z^3 - 3wz^2 + 8 = 0 \), where \( w \) stands for \( -\sqrt[3]{4} \). First, \( z \) was replaced by \( u + \frac{1}{2}w \), which produced the equation \( u^3 + 3w^2u + 5 = 0 \), whose discriminant is \( d = 6 \). Next the values \( p = -\sqrt[3]{\frac{5}{2} - \sqrt{6}} \) and \( q = -\sqrt[3]{\frac{5}{2} + \sqrt{6}} \) were calculated. Finally, the solutions were written down as shown on page 223.

Problems

7. Find three roots (in non-decimal form) for each of the following:

(a) \( z^3 - 6z + 4 = 0 \)  (b) \( w^3 - 3w^2 - 9w + 23 = 0 \)  (c) \( w^3 + w^2 + w + 1 = 0 \)
Dynamic Fractals

For those who are interested in the programming of fractal images, here is a BASIC program that will produce the cube-roots-of-8 image. It can easily be adapted to work in other languages (and on other fractal examples).

```basic
100 hpix = 639 : vpix = 479 : left = -3 : right = 3 : low = -3 : high = 3
120 window (left, high) - (right, low)
130 hdel = (right - left)/hpix : vdel = (high - low)/vpix
200 for j = 0 to hpix
205 p = left + j * hdel
210 for i = 0 to vpix
215 q = low + i * vdel
220 x = p : y = q : col = 0
300 for k = 1 to 200
310 denom = x * x + y * y
312 if denom < 0.0001 then 400 else denom = denom * denom
320 newx = (2 * x + 8 * (x * x - y * y)/denom)/3
322 y = (2 * y - 16 * x * y/denom)/3
330 x = newx
340 if 0.5 < abs(x - 2) + abs(y) then 350
345 col = 1 : goto 400
350 if 0.5 < abs(x + 1) + abs(y - 1.73) then 360
355 col = 2 : goto 400
360 if 0.5 < abs(x + 1) + abs(y + 1.73) then 370
365 col = 3 : goto 400
370 next k
400 pset(p, q), col
500 next i
510 next j
600 end
```

Line 100 matches the calculations to the resolution of the host computer screen, which is assumed to have the standard VGA dimensions of 640 pixels by 480 pixels. Lines 200 and 210 instruct the computer to examine every one of the $640 \times 480 = 307200$ pixels, which in line 120 were associated with the rectangular region defined by $-3 \leq x \leq 3$ and $-3 \leq y \leq 3$. This viewing window contains all three roots with a little room to spare. At any moment, the pixel being examined represents the point $(p, q)$. Initially, the pixel is assigned the color 0 (the background color); this will be changed to a different color if the orbit of this pixel wanders close enough to one of the three target points. Taxicab distance is used to check proximity, for millions of such time-consuming checks will have to be made. Iteration for a given pixel halts as soon as a color change is made, or if 200 iterations are reached (very unlikely). The active point on the orbit is named $(x, y)$. Color decisions are made in lines 340 through 365. The actual marking of the colored dot is at line 400.
Dynamic Fractals

1. One of the distinguishing aspects of Newton-Raphson fractals is that \( \infty \) is a repelling fixed point of the root-finding process, and is therefore a member of the Julia set. Explain this observation, and draw a contrast with item 7 on page 9.

2. Given a seed value \( z_0 \), a recursively defined sequence \( z_n = f(z_{n-1}) \) is sometimes called the orbit of \( z_0 \). Whether or not it converges, an orbit is always an invariant set of points. Explain this remark.

3. Is it possible for an orbit to consist of only finitely many different points? If so, how many points are possible?

4. In Figure 3, the basin of attraction of the root \( z = 2 \) has infinitely many components arranged along the negative real axis. Show that the sizes of these components increase approximately according to a geometric progression as they recede from the origin.

5. Figure 5 shows a Julia set that corresponds to a Newton-Raphson process. What do you think the process is?

6. In the cube-root example shown on page 221, the Julia set contains the ancestral tree of 0 (which can actually be thought of as the ancestral tree of \( \infty \)). Does this infinite set of points fill the Julia set, or are there other points to be accounted for?

7. Let \( N(z) = \frac{1}{3}(2z + 8z^{-2}) \) be the cube-roots-of-8 function, and show that

\[
N(z \text{cis}(120)) = N(z)\text{cis}(120).
\]

What does this tell you about the appearance of Figure 3? In particular, what does this tell you about the orbits of \( N \)?

8. Looked at with a magnifying glass, the Julia set shown in Figure 3 has the same appearance, no matter which ancestor of 0 the glass is focused on. Explain this remark, and account for the phenomenon.

9. Recall the slow square-root finder \( f(z) = \frac{z + m^2}{z + 1} \), which was introduced on page 9, and which appears in Figure 2. When \( m^2 = 3 + 4i \), this process has only one attractor; the other root goes undetected. Is this true for all values of \( m \)? Is there a way to modify the definition of \( f \) so that the other root becomes the attractor?
Dynamic Fractals

1. Let \( f(z) = \frac{1}{2}(z + m^2z^{-1}) \) be the Newton-Raphson square-root finder, and consider the following proposition:

For any seed value \( z_0 \) that is closer to \( m \) than it is to zero (in other words, \( |z_0 - m| < |z_0| \)), the orbit \( \{z_n\} \) approaches \( m \), so that the errors \( |z_n - m| \) diminish monotonically to zero.

Consider also the following proof:

Calculate

\[
|z_{n+1} - m| = |f(z_n) - m| = \left| \frac{1}{2} \left( z_n + \frac{m^2}{z_n} \right) - m \right|
\]

\[
= \left| \frac{(z_n - m)^2}{2z_n} \right| = \frac{|z_n - m|^2}{2|z_n|}.
\]

Because one can inductively assume that \( |z_n - m| < |z_n| \), it follows that

\[
|z_{n+1} - m| = \frac{|z_n - m|^2}{2|z_n|} = \frac{|z_n - m|}{|z_n|} \frac{|z_n - m|}{2} < \frac{|z_n - m|}{2},
\]

thus each error is less than half of the preceding error. It follows that the errors approach 0 at least as fast as \( \left( \frac{1}{2} \right)^n |z_0 - m| \) does.

Consider also the following data, which pertains to the ongoing example \( m = 2 + i \), and which was generated by the seed value \( z_0 = -25 + 53i \):

<table>
<thead>
<tr>
<th>( x_n )</th>
<th>( y_n )</th>
<th>( e_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-25.00000000</td>
<td>53.00000000</td>
<td>58.59180830</td>
</tr>
<tr>
<td>-12.48005242</td>
<td>26.46228888</td>
<td>29.29163827</td>
</tr>
<tr>
<td>-6.20006789</td>
<td>13.15561500</td>
<td>14.66288134</td>
</tr>
<tr>
<td>-3.01960728</td>
<td>6.42588372</td>
<td>7.39166229</td>
</tr>
<tr>
<td>-1.34471031</td>
<td>2.90193140</td>
<td>3.84765254</td>
</tr>
<tr>
<td>-0.30216939</td>
<td>0.76253020</td>
<td>2.31438454</td>
</tr>
<tr>
<td>1.44206660</td>
<td>-2.21719142</td>
<td>3.26521214</td>
</tr>
<tr>
<td>0.39635530</td>
<td>-0.22089279</td>
<td>2.01550379</td>
</tr>
<tr>
<td>0.94006174</td>
<td>5.34899072</td>
<td>4.47629192</td>
</tr>
<tr>
<td>0.88053782</td>
<td>2.46621343</td>
<td>1.84471607</td>
</tr>
<tr>
<td>1.35214393</td>
<td>0.95046307</td>
<td>0.64974718</td>
</tr>
</tbody>
</table>

The data is inconsistent with the proposition. Explain. Now what do you think of the “proof” that appears above? Can you find the flaw?
Dynamic Fractals

Quadratic functions.

Now that you have followed the root-finding route to chaos and fractals, the time has come to consider other examples. There being little promise of interesting results when linear functions are iterated, it seems logical to begin the next phase of your investigation with the humble quadratic function. Therefore, first examine \( Q(z) = z^2 \), viewed as a dynamic system.

1. Show that \( Q \) has two fixed points, one attracting and one repelling. You will also find it helpful to include \( \infty \) in your catalogue of interesting points — it is fixed and attracting.

2. Show that the Julia set for \( Q \) is the unit circle.

3. Show that \( Q \) has exactly one 2-cycle.

4. Show that \( Q \) has exactly two 3-cycles.

5. Suppose that \( 2 \leq k \). Explain why all \( k \)-cycles of \( Q \) belong to the Julia set.

6. How many 4-cycles does \( Q \) have?

7. Show that \( Q \) has \( k \)-cycles of all lengths.

8. Find all the ancestors of \( z = 1 \). There are infinitely many of them, and the Julia set contains them all. These points are called eventually fixed.

9. Give an example of a \( z \)-value that is not part of a 4-cycle for \( Q \), but whose orbit eventually becomes a 4-cycle.

10. You have now identified many specific points that are found in the Julia set of the function \( Q \). Are there others?

11. The fixed point at the origin is not only an attracting fixed point of \( Q \), it is super-attracting. Explain this remark. What can you say about the rate at which sequences \( z_n = Q(z_{n-1}) \) approach the origin?

12. Let \( z_0 = \text{cis}(\theta) \), where the angle \( \theta \) is randomly chosen between 0 and 360. Calculate several terms of the orbit \( \{z_n\} \). Is the result what you expected?

13. What can you say about the ancestral tree of a \( z \)-value that is not in the Julia set?

This is an atypical dynamic system, for there are very few Julia sets that can be drawn accurately by hand. Moreover, as was the case with the Newton-Raphson square-root finder, the geometric simplicity of this Julia set makes its members look indistinguishable, even though they exhibit a wide variety of behaviors.

14. Verify that linear functions \( f(z) = mz + b \) are indeed rather tame when viewed as dynamic systems. For what \( m \) and \( b \) is there an attractor?
Dynamic Fractals

Infinity and the complex sphere.

Although infinity can not be given full-fledged numerical status — it does not have real and imaginary parts, for one thing — this concept plays an essential role in the complex number system, especially when functions are being viewed as transformations. From a geometric point of view, adjoining infinity causes the complex number system to take on a pleasing spherical form.

There are many ways of associating the points of a sphere with complex numbers, but the following is probably the most common: Given a complex number $z = a + bi$, consider the line through $(a, b, 0)$ and $(0, 0, 1)$. This line intersects the sphere $p^2 + q^2 + r^2 = 1$ twice — at $(0, 0, 1)$ and at a second point that is called the stereographic projection of $z$. For example, $1 + i$ is mapped to $(\frac{2}{3}, \frac{2}{3}, \frac{1}{3})$. In effect, the flat (equatorial) complex plane is wrapped around the sphere, in such a way that the north pole $(0, 0, 1)$ is the one spherical point that is not matched with any complex number. Moreover, points that are far from the origin in the complex plane are associated to points on the sphere that are near the north pole. Therefore, it makes sense to think of $(0, 0, 1)$ as $\infty$.

Meanwhile, the south pole $(0, 0, -1)$ corresponds to zero, the equator of the sphere coincides with the circle of unit complex numbers, and the southern hemisphere is identified with the interior of the unit circle.

1. Find equations that allow you to calculate $p$, $q$, and $r$, given values for $a$ and $b$. Inversely, find equations that allow you to calculate $a$ and $b$, given values for $p$, $q$, and $r$.

2. Locate the points on the sphere that correspond to 2 and to $\frac{1}{2}$, to $i$ and $-i$, to $2 + i$ and $\frac{2}{3} - \frac{i}{3}$, and to $-1 + i$ and $-\frac{1}{2} - \frac{1}{2}i$. Do these pairs of points suggest a pattern?

3. One of the interesting geometric properties of stereographic projection is that straight lines in the complex plane are mapped to circles on the complex sphere, while circles in the complex plane are also mapped to circles on the complex sphere. Verify that this is true. In particular, what spherical point do all straight lines share?

4. Many complex functions become easier to visualize when the spherical model is used. For example, show that the function $R(z) = z^{-1}$ is represented as a 180-degree rotation of the sphere. The most striking aspect of this interpretation is that the function is everywhere continuous!

5. What complex function is represented by the 90-degree spherical rotation — about the north-south axis — that takes 1 onto $i$?

6. Like most functions, stereographic projection introduces distortion, especially near $\infty$. Although distances may be drastically altered, however, angles are not. By considering some examples, convince yourself of this so-called conformal property of stereographic projection. (By the way, the preservation of angular size is a central theme in the theory of complex functions.)
Dynamic Fractals

1. Suppose that the function $f$ has the strange property that every seed value $z_0$ produces a 2-cycle. What does this tell you about $f$? Can you think of any examples of such functions?

2. This exercise is meant to emphasize the role of the derivative in describing locally how a function transforms the complex plane. Return to Figure 3 (the cube-roots-of-8 diagram), and recall that $-\sqrt[3]{4}$ is the immediate ancestor of 0 that is real. Let $w$ be the first-quadrant immediate ancestor of $-\sqrt[3]{4}$ (there are three in all). It is convenient to express $w$ in polar form as $1.1811\text{cis}(78.12)$. As a first step, verify this approximation, using the coordinates on page 223 (item 1).

The Newton-Raphson function $N(z) = \frac{1}{3}(2z + 8z^{-2})$ transforms $w$ and its immediate neighborhood onto a neighborhood of $-\sqrt[3]{4}$. In the process, a certain amount of local stretching and turning takes place, as Figure 3 shows. To quantify this stretching and turning, calculate $N'(w)$, putting your answer into the useful polar form. Remember that $N'(w)$ relates small differences $w - z$ to correspondingly small differences $N(z) - N(w)$.

3. Given a seed value $a_0$ between 0 and 1, define a sequence recursively by the following rule for positive $n$:

$$a_n = \begin{cases} 2a_{n-1} & \text{if } a_{n-1} < 0.5 \\ 2a_{n-1} - 1 & \text{otherwise} \end{cases}$$

(a) Calculate $a_5$, given that $a_0 = \frac{1}{4}\pi$.
(b) Choose $a_0$ so that the sequence $\{a_n\}$ is eventually constant.
(c) Choose $a_0$ so that the sequence is a 5-cycle.

4. Refer to the preceding item. Is it possible to choose $a_0$ so that the resulting sequence is aperiodic? This means that the sequence is neither periodic nor eventually periodic. Could such a sequence be produced on the computer (or on a calculator)?

5. Consider the squaring function $Q(z) = z^2$.
(a) Show that there are seed values $z_0$ whose orbits never meet the real axis.
(b) The ancestral tree of 1 is a dense subset of the unit circle (i.e., the Julia set). Formulate a definition of what this means.
(c) Give an example of an infinite subset of the unit circle that is not dense.
(d) Give another example of a dense subset of the unit circle that has no points in common with the ancestral tree of 1.
(e) Show that it is impossible to predict the future for sequences that are seeded in the vicinity of 1.

6. It is occasionally suggested that complex sequences that approach zero do so by spiralling toward zero. This is not always an accurate description, however. Consider the following two examples, one linear, the other quadratic:
(a) $z_0 = 1$ and $z_n = 0.5iz_{n-1}$
(b) $z_0 = 0.9\text{cis}(45)$ and $z_n = z_{n-1}^2$
Dynamic Fractals

1. Although degree measure and radian measure are familiar ways to describe angles and locate points on circles, it will be convenient for you to use the revolution as the unit of angle measure. In this system, a right angle has measure \( \frac{1}{4} \), the angles of an equilateral triangle all have measure \( \frac{1}{6} \), and points on a circle can be described by numbers between 0 and 1.

For points on the unit circle, the squaring function \( Q \) acts by doubling the polar angle. Consider the points described by rational angles (for example, \( \frac{5}{12} \) revolution). Prove that repeated squaring reveals any such angle to be either periodic (such as \( \frac{2}{7} \)) or eventually periodic (such as \( \frac{5}{12} \)). How can you tell which is which? Among the eventually periodic examples, which ones are eventually fixed?

2. Consider the dynamic system defined in item 3 on page 19, and let \( a_0 = \frac{1}{2} \sqrt{2} \). The next four terms of the orbit are \( a_1 = \sqrt{2} - 1, a_2 = 2\sqrt{2} - 2, a_3 = 4\sqrt{2} - 5, \) and \( a_4 = 8\sqrt{2} - 11 \). Prove that this sequence is infinite (i.e., aperiodic), by showing that no term appears for a second time.

3. Show that the preceding result applies to any irrational \( a_0 \).

4. Now that you have examined the simple quadratic system described by \( Q(z) = z^2 \), the time has come to consider the rest of the quadratic family. It suffices to look at the Julia sets for the examples \( Q_c(z) = z^2 + c \), because other quadratic Julia sets exactly duplicate these. Consider \( Q_{-5/16} \). Before you try to draw this Julia set (an exceedingly complicated task), find out a few things about it:
   (a) Find the two finite fixed points of \( Q_{-5/16} \) and show that one is attracting and the other is repelling.
   (b) The repelling fixed point is part of the Julia set — why? Where is the attracting finite fixed point, and why?
   (c) Calculate three members of the ancestral tree of the repelling fixed point. Where are these numbers found?

5. Verify that the fixed points for \( Q_c(z) = z^2 + c \) are

\[
z = \frac{1}{2}(1 + \sqrt{1 - 4c}) \quad \text{and} \quad z = \frac{1}{2}(1 - \sqrt{1 - 4c}).
\]

6. Investigate the dynamic properties of \( Q_c \) when \( c = \frac{1}{8}(-2 + 3i) \). One of the fixed points is \( \frac{1}{4}(-1 + i) \). Find the other one, and decide whether either fixed point is attracting.

7. Find the value of \( c \) for which \( Q_c \) has \( \frac{1}{2}i \) as a fixed point. For this value of \( c \), what is the other finite fixed point? Is either finite fixed point attracting?
Dynamic Fractals

Figure 6 shows the filled-in Julia set for \( Q_{-5/16}(z) = z^2 - \frac{5}{16} \). Its boundary is the actual Julia set. The black points generate sequences (orbits) that converge to the attracting fixed point, which is \( z = -\frac{1}{4} \). White points, on the other hand, lead to the superattracting fixed point at \( \infty \). The boundary points form the invariant Julia set, whose members exhibit various forms of indecisive behavior when \( Q_{-5/16} \) is applied to them. The prominent point at the extreme right is the repelling fixed point \( z = \frac{5}{4} \). The other most conspicuous Julia points are ancestors of this fixed point. The leftmost point is \( z = -\frac{5}{4} \), which is the immediate ancestor of the repelling fixed point. Two generations back, notice the extremes \( z = \pm \frac{1}{4} i \sqrt{15} \approx \pm 0.968246i \).

Figure 7 also shows the filled-in Julia set for \( Q_{-5/16}(z) = z^2 - \frac{5}{16} \), but the basin of attraction for \( \infty \) is divided into bands by a nested system of closed curves, which represent convergence to the attractor at \( \infty \). These curves can be described recursively. First the circle \( |z| = 2 \) was selected; it shows in part at the corners of the diagram. Next the ancestors of its points form a smaller oval, seen in its entirety. The ancestors of the points on the oval form the next curve, and so on. The curves approach the Julia set.

1. The circle \( |z| = 2 \) was chosen for the following reason: If \( z_0 \) is any seed value that satisfies \( 2 < |z_0| \), then the resulting orbit \( \{z_n\} \) approaches \( \infty \). Prove this, thereby showing that the Julia set for \( Q_{-5/16} \) is contained completely inside the circle.

2. Figure 7 makes it clear that the oval that is ancestral to the circle \( |z| = 2 \) is not itself a circle. Is it an ellipse, however?

3. Do the points on successive curves correspond in a one-to-one fashion?

You will soon examine some of the less conspicuous members of this Julia set.
Dynamic Fractals

1. Use Figure 7 to help you answer this question: If \( z_0 \) is a seed value whose orbit \( \{z_n\} \) approaches \( \infty \), need it be true that \( |z_n| \) grows monotonically?

2. Any function \( Q_c \) has a single 2-cycle. Find formulas for its members, similar to those in item 5 on page 20. Hint: the two fixed points (the 1-cycles) are also solutions to the fourth-degree equation \( Q_c^2(z) = z \). Use the special case \( c = 0 \) to check your calculations.

3. Explain why \( Q_c \) must have two 3-cycles, no matter what \( c \) is. Do not attempt to find explicit formulas for the members of the 3-cycles, however.

4. For what values of \( c \) does \( Q_c \) have a superattracting fixed point?

5. For what values of \( c \) does \( Q_c \) have an attracting fixed point? Find as many examples as you can.

6. Given a quadratic function \( Q_c \), show that
   (a) the ancestors of any point \( z \) are opposites (the origin is the midpoint of the segment that joins them);
   (b) the average of the finite fixed points is \( \frac{1}{2} \);
   (c) it is not possible for both of the finite fixed points to be attracting;
   (d) the Julia set of \( Q_c \) has odd symmetry (180-degree rotational symmetry about the origin).

7. Show that \( p \) is a fixed point of \( Q_c \) if and only if \( c = p - p^2 \).

8. If \( p \) is a fixed point of \( F \), and if \( |F'(p)| = 1 \), then \( p \) is called an indifferent fixed point. Make up real-variable examples that show how
   (a) an indifferent fixed point can behave as though it were attracting, meaning that all nearby seed points generate sequences that approach \( p \);
   (b) an indifferent fixed point can behave as though it were repelling, meaning that all nearby seed points generate sequences that are driven away from \( p \).

9. Show that \( Q_c \) has an attracting fixed point if \( c = -\frac{2}{3} \), and two repelling fixed points if \( c = -1 \). What if \( c = -\frac{3}{4} \)?

10. If \( p \) is a fixed point of \( F \), with \( F'(p) = \frac{1}{2}i \), and if \( z_0 \) is a point near \( p \), then what happens when you apply \( F \) repeatedly to \( z_0 \)? In other words, what does the orbit of \( z_0 \) look like?

11. Given a dynamic system, a set of points is called completely invariant if it contains the orbits and ancestral trees of all of its members. Explain why Julia sets are completely invariant.
Dynamic Fractals

The dyadic ruler.

Figure 8 shows yet another view of the filled-in Julia set for $Q_{-5/16}(z) = z^2 - \frac{5}{16}$. The dynamic squaring process has been further highlighted by the addition of dyadic field lines, which cross the escape contours perpendicularly. Notice how each field line lands on the Julia set at one of the eventually fixed points (ancestors of the repelling fixed point). These points are naturally indexed with rational numbers whose denominators are powers of 2, which is why they are called dyadic rationals. Just as with an ordinary ruler, these markings enable you to find other Julia points of interest.

1. Use the built-in ruler to find the Julia points whose dynamic indices are $\frac{1}{3}$ and $\frac{2}{3}$. What is special about these points?

2. Use the ruler to find the immediate ancestors of the points in the preceding item.

3. Explain how to locate the Julia point whose index is $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots = \sum_{n=0}^{\infty} (-\frac{1}{2})^n$.

4. Describe the workings of a computer program that would produce Figure 8.
Dynamic Fractals

Self-similarity.

Figure 10 shows four magnifications of the Julia set in the vicinity of the Julia point with dynamic index \( \frac{1}{3} \) (one of the 2-cycle points). The frame widths are 0.4, 0.04, 0.004, and 0.0004, reading clockwise from the upper left corner. The 2-cycle point is at the center of each frame. It can be calculated by inserting \( c = -\frac{5}{16} \) into the formula in item 2 on page 234; the result is \( z = \frac{1}{4}(-2 + i\sqrt{7}) \), which is approximately \(-0.5 + 0.6614378i\).

1. It should be evident that the focus of this zoom sequence (the 2-cycle point) is indeed extreme, found at the bottom of a valley whose silhouette changes imperceptibly with each tenfold magnification. Explain why similar pictures would be obtained if you focused your microscope on any point that is eventually of period 2.

2. Consider the function

\[
F(z) = Q_{-\frac{5}{16}}^2(z) = \left(z^2 - \frac{5}{16}\right)^2 - \frac{5}{16}.
\]

What are the fixed points of \( F \)? For each fixed point \( p \) of \( F \), calculate \( F'(p) \). How does the Julia set of \( F \) compare with the Julia set of \( Q_{-\frac{5}{16}} \)?

3. Suppose that \( F(p) = p \) and that \( F'(p) = 2 - i \). Explain why \( p \) is in the Julia set of \( F \). Suppose that \( z_0 \) is very close to \( p \) — make a sketch of what the first few terms of the orbit \( \{z_n\} \) would look like.

4. What are the indices of the two immediate ancestors of the quadratic Julia point indexed by \( \frac{7}{11} \)?

5. How many 6-cycles does the function \( Q_0 \) have? What about 6-cycles for \( Q_c \) in general?

6. There is one \( c \)-value for which the 2-cycle of \( Q_c \) is exceptional (see item 2 on page 234). Find it and explain its significance. In particular, is there anything exceptional about the Julia set for this \( c \)-value?

7. Suppose that \( p \) and \( q \) are complex numbers whose sum is \(-1\). Show that \( p \) and \( q \) form the 2-cycle for some \( Q_c \). In other words, find \( c \) in terms of \( p \) and \( q \).
Dynamic Fractals

1. Find an approximation $m/n$ for $\sqrt{10}$ that is accurate to five decimal places, with $m$ and $n$ positive integers and $n < 250$. Use a calculator to check your accuracy.

2. Give a specific example of a 4-cycle $z_0, z_1, z_2, z_3, z_0, \ldots$ for the function $Q_0(z) = z^2$.

3. Show that it is possible for a quadratic function $Q_c$ to have only one finite fixed point.

4. The figure at right shows the Julia set of the function $Q_c$ in the case $c = -\frac{39}{100}$. Find the axis intercepts, to four-place accuracy.

5. In polar form, find all the cube roots of $8i$.

6. Give three examples of infinite sets that are left invariant by the function $Q_0(z) = z^2$.

7. Let $f(z) = \frac{5}{2}z(1 - z)$. Find the fixed points of $f$, then classify each as either attracting or repelling, with justification for your choices.

8. Consider the function

\[ G(z) = \frac{1}{2} \left( z + \frac{8 + 6i}{z} \right). \]

(a) Given that $z_0 = i$ and $z_{n+1} = G(z_n)$, find the limit of $z_n$ as $n \to \infty$.

(b) Find a nonzero $z_0$ for which the resulting sequence does not approach a limit.

9. Figure 11 shows the Julia set of a Newton-Raphson root-finder. Write a specific function that could have produced this figure.

10. Find coordinates for the 2-cycle in the Julia set of item 4.

11. Consider the real-valued function

\[ f(x) = \frac{1}{3} \left( 2x + \frac{8}{x^2} \right), \]

and let $x_0 = 3$. Prove that the sequence defined recursively by $x_{n+1} = f(x_n)$ is monotonically decreasing.
Dynamic Fractals

An amazing zoom.

Figure 12 presents a short sequence of tenfold magnifications of the Julia set for $Q_{-5/16}$. This time, the magnification windows are centered at $0.72071430422 + 0.60916632028i$, which is (approximately) one of the six 3-cycle points. The initial frame, whose width is 0.3, suggests that this point is located on the side of a gentle hill. The next two frames suggest that the hill is actually rather steep. The fourth frame (leftmost in the second row) suggests that the point is on the face of a cliff. The last five frames suggest that you are actually looking at the underside of an overhang. The final frame has magnified a small piece of the original Julia set enormously — Figure 8 would be more than 86000 miles across if it could now be seen in its entirety.

The sequence of magnifications in Figure 12 can be continued forever, at least in your imagination. It is evident that the focal point of this zoom — rather than resting on the edge of a simple contour — is in fact surrounded by a spiralling Julia set. If it could be drawn, the field line of index $\frac{1}{7}$ would have to spiral infinitely many times in order to reach the Julia set. This zoom sequence also shows that most dyadic field lines must spiral as well — though at at most finitely many times — in order to reach their eventually fixed Julia points.

This is a startling demonstration that the points of a Julia set are not at all alike! Moreover, the incredible magnification required makes it impossible to ever see more than just a little bit of the detail just described.

1. Figure 12 shows that the field line of index $\frac{1}{7}$ reaches the Julia set of $Q_{-5/16}$ only after spiralling infinitely many times. What other field lines will behave in the same fashion as they approach this Julia set?

2. Explain how the infinite geometric series $\frac{1}{8} + \frac{1}{64} + \frac{1}{512} + \ldots$ can be used to locate (approximately) one of the points of period 3 in the Julia set of $Q_{-5/16}$. 

May 2006

Phillips Exeter Academy
1. Let $p$ and $q$ denote the 2-cycle points (the points of period 2, that is) for $Q_c$. Calculate $Q'_c(p)Q'_c(q)$, in terms of $c$ (see item 2 on page 234). What is the significance of this product? What is special about the examples $c = -1 + \frac{1}{8}i$ and $c = -1$?

2. Drawing pictures of Julia sets can take a long time, even for a powerful computer. Given a seed value $z_0$, in other words, it can take many iterations of $Q_c$ before it is possible to be sure that the orbit $\{z_n\}$ has committed itself to infinity (at which time a color is assigned to $z_0$). This decision is made once $|z_n|$ has grown large enough. Prove the following:

(a) If $|c| < |z|$ and $2 < |z|$, then $|z| < |Q_c(z)|$.
(b) If $|c| < |z_m|$ and $2 < |z_m|$, then $|z_n|$ increases monotonically to infinity for $m \leq n$.

This shows that, for any $c$-value smaller than 2 in magnitude, the circle $|z| = 2$ is large enough to use as an escape threshold, for the Julia set is contained safely inside. For larger $c$-values, the circle of radius $|c|$ will do.

3. Figure 13 shows the filled-in Julia set for $Q_{-1}$. The width of the frame is about 3.5, and it is centered at the origin. Calculate the fixed points and the 2-cycle for this system, then explain why the 2-cycle is not part of the Julia set, while both fixed points are. Notice that this Julia set has an infinite number of conspicuous points that are each indexed twice. Can you identify these points and their indices?

4. Prove that the 2-cycle for $Q_c$ is attracting (what does that mean for a cycle?) whenever $|c+1| < \frac{1}{4}$. For such a value of $c$, the 2-cycle is therefore not part of (it is surrounded by) the Julia set.

5. Consider the functions

$$T(z) = \frac{z - m}{z + m} \quad \text{and} \quad N(z) = \frac{1}{2} \left( z + \frac{m^2}{z} \right).$$

Recall that $N$ is the Newton-Raphson square-root finder that has $m$ as a superattracting fixed point. Verify that $T$ is a one-to-one transformation of the complex sphere, and that $Q_0(T(z)) \equiv T(N(z))$. This shows that $T$ sets up a dynamic equivalence between the Newton-Raphson system and the quadratic system defined by $Q_0$. Notice that $T(m) = 0$ matches two fixed points. What about the others? What about the Julia sets?
Dynamic Fractals

1. Reconsider the transformation of coordinates $T$ introduced in item 7 of page 28:
   (a) Show that $T$ is an invertible function, by finding a formula for $T^{-1}$.
   (b) What are $T(\infty)$ and $T^{-1}(\infty)$?
   (c) Each point of the unit circle is mapped by $T^{-1}$ to a point of the Julia set of the
      Newton-Raphson function $N$. Using the familiar case $m = 2 + i$ as an illustration, find
      explicit coordinates for $T^{-1}(i)$.
   (d) As in the preceding, use this correspondence to find the 2-cycle for $N$.
   (e) Use $T$ to reason that any Newton-Raphson orbit $\{z_n\}$ converges to the attractor that
      $z_0$ is closest to.

2. Figure 14 shows the filled-in Julia set for $Q_c$ in the special case $c = \frac{1}{4} + \frac{1}{2}i$. The frame
   width is about 3.
   (a) Calculate the fixed points of $Q_c$, and find them in the picture. One point is repelling;
      the other point is indifferent.
   (b) Use item 3 on page 237 to explain the appearance of the Julia set near the repelling
      fixed point.
   (c) Find the ancestors of the repelling fixed point in the picture. What do the dyadic field
      lines do as they approach these members of the Julia set?
   (d) Use $Q'_c$ to explain the appearance of the Julia set near the indifferent fixed point.

3. The table below lists two of the three 4-cycles for the quadratic function $Q_{-5/16}$, whose
   filled-in Julia set appears in Figure 8.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.08184165</td>
<td>0.27723942</td>
<td>0.27974265</td>
<td>0.81205572</td>
</tr>
<tr>
<td>0.78101966</td>
<td>0.59985830</td>
<td>-0.89367854</td>
<td>0.45433323</td>
</tr>
<tr>
<td>-0.06233826</td>
<td>0.93700225</td>
<td>0.27974265</td>
<td>-0.81205572</td>
</tr>
<tr>
<td>-1.18658715</td>
<td>-0.11682219</td>
<td>-0.89367854</td>
<td>-0.45433323</td>
</tr>
</tbody>
</table>

   The dyadic indices for the left-hand column are $\frac{1}{15}$, $\frac{2}{15}$, $\frac{4}{15}$, and $\frac{8}{15}$. For the right-hand
   column, they are $\frac{1}{5}$, $\frac{2}{5}$, $\frac{4}{5}$, and $\frac{3}{5}$. The field lines for the points in one of these cycles will
   have to spiral infinitely many times (as in Figure 12) in order to land on the Julia set, while the field lines for the points in the other cycle will not have to spiral. Which cycle is which?
   There is a third 4-cycle, not tabled. What are the coordinates of its members? What are
   their dyadic indices, and what do their field lines look like?

4. As $c$ increases in magnitude, the attracting fixed point $p$ of $Q_c$ drifts away from the
   origin, which diminishes its strength of attraction. For certain values of $c$, the strength
   diminishes enough to make $p$ indifferent. These $c$-values form a simple curve that surrounds
   the origin — find an equation for this curve. Remember that $c$ is expressed in terms of $p$
   by the equation $c = p - p^2$.
Dynamic Fractals

1. Figure 16 is a slight variation on the third frame of Figure 10. The center of the window has been shifted up and right, moving the 2-cycle point closer to the lower left corner. This brings one more large peak into view, and gives you the opportunity to verify experimentally that the derivative of $Q_c^2$ at this 2-cycle point is indeed

$$4(1 + c) = 4 \left( 1 - \frac{5}{16} \right) = \frac{11}{4},$$

a real value. (See item 1 on page 27.) Thus $Q_c^2$ moves all the points in this frame radially away from the 2-cycle point, expanding distances by approximately $\frac{11}{4}$. Use a ruler to confirm this.

2. In order for a cycle to be superattracting, the product of all values of $Q'_c$ along the cycle must be zero. This says simply that zero must belong to the cycle. This makes it easy to search for superattracting cycles — just seed an orbit with 0 and request that it close. For example, to find the superattracting 2-cycle, first write out a few terms of the orbit of 0,

$$0, c, c^2 + c, (c^2 + c)^2 + c, \ldots,$$

then solve $c^2 + c = 0$. This equation is quadratic — what is the significance of its two solutions? What equation do you solve to find those $Q_c$ that have superattracting 3-cycles? How many examples are there?

3. Figure 17 shows a very small neighborhood of the right-hand fixed point for $Q_{-1}$, whose filled-in Julia set appears in Figure 13. With the help of a ruler, comment on the sizes and placements of the visible black components of the filled-in Julia set. How does the derivative enable you to predict what you see?

4. If the preceding question had featured the other fixed point, would there have been any significant changes in your answer?
Dynamic Fractals

The Koch Snowflake is one of the best-known fractals. It was also one of the earliest to appear, invented by Helge von Koch in 1904. This was before the birth of Benoit Mandelbrot, who coined the term fractal to describe curves, like the snowflake, that are so wrinkly that they can not be considered to be one-dimensional. Although this curve is not a Julia set (it is not associated with any dynamic system), its appearance and subtle properties will remind you of features you have seen in Julia sets.

The curve is defined as the limit of a sequence of piecewise-linear curves: Stage 0 consists of an equilateral triangle. Stage 1 is obtained from stage 0 by replacing the middle third of each edge by a pair of segments, arranged so that a small equilateral triangle protrudes from the edge. In general, stage \( n + 1 \) is obtained from stage \( n \) by replacing the middle third of each of the \( 3 \cdot 4^n \) edges of stage \( n \) by a pair of segments, arranged so that a small equilateral triangle protrudes from the edge. The figure at right shows the completion of stage 4, a 768-segment curve.

To be specific, suppose that the process is applied to the triangle \( ABC \), where \( A = (0, 0) \), \( B = (3, 0) \), and \( C = \left( \frac{3}{2}, -\frac{3}{2} \sqrt{3} \right) \). Notice that each stage of the process includes some points that persist, certain to be part of the limit curve. For example, the first stage includes the acute point \( \left( \frac{3}{2}, \frac{1}{2} \sqrt{3} \right) \) and the obtuse point \( (1, 0) \). Like \( \left( \frac{5}{4}, \frac{1}{4} \sqrt{3} \right) \) in stage 1, however, most points are temporary — certain to disappear.

1. Verify that stage \( n \) consists of \( 3 \cdot 4^n \) segments.
2. Stage 1 has six acute points: the original \( A \), \( B \), and \( C \), plus \( \left( \frac{3}{2}, \frac{1}{2} \sqrt{3} \right) \), \( (0, -\sqrt{3}) \), and \( (3, -\sqrt{3}) \). Stage 1 also has six obtuse points: \( (1, 0) \), \( (2, 0) \), \( \left( \frac{5}{2}, -\frac{1}{2} \sqrt{3} \right) \), \( (2, -\sqrt{3}) \), \( (1, -\sqrt{3}) \), and \( \left( \frac{1}{2}, -\frac{1}{2} \sqrt{3} \right) \). Show that stage \( n \) has \( 4^n + 2 \) acute points and \( 2 \cdot 4^n - 2 \) obtuse points in general. (Hint: What is the sum of the exterior angles of the figure?)
3. Not all persistent points are as conspicuous as acute and obtuse points are. For example, \( \left( \frac{2}{3}, 0 \right) \) persists. Find it in the figure, and prove that it is part of the limit curve.
4. To reach most snowflake points from the exterior of the snowflake, one must follow paths that spiral many times before they land. Explain why.
5. The points of any stage can be indexed continuously from 0 to 1, starting and finishing at \( A \), so that \( C \) gets index \( \frac{1}{3} \) and \( B \) gets \( \frac{2}{3} \). Show that such indexing can be done in a piecewise-linear fashion, so that persistent points receive the same index at every stage.
6. Some snowflake points are pure limit points, which need not appear in any stage. For example, show that \( \left( \frac{15}{14}, \frac{3}{14} \sqrt{3} \right) \) is the limit of a geometric sequence of acute points whose first term is \( B \) and whose second is \( \left( \frac{3}{2}, \frac{1}{2} \sqrt{3} \right) \). (Hint: The sequence is conveniently expressed in terms of a geometric sequence of complex numbers.)
Dynamic Fractals

1. Verify that the curve derived in item 4 on page 243 really is a cardioid. Start with the polar graph \( r = 1 - \cos \theta \), reduce it by one half, then shift the graph to the right by \( \frac{1}{4} \).

2. Let \( c \) be the value that results when \( \theta = 120 \) is substituted into the cardioid equation on page 243. By considering the behavior of \( Q_c \) at its indifferent fixed point, predict the appearance of the Julia set for \( Q_c \).

3. The preceding question can be repeated for many other values of \( c \) chosen from the cardioid. Sample a few of them, concentrating on the rational values of \( \theta \).

4. What is special about numbers of the form \( c = -1 + \frac{1}{4} \operatorname{cis} \theta \)? In particular, what happens if \( \theta \) is chosen to be a rational angle, such as 90, 120, 180, 216, or 330?

5. Figure 20 shows the filled-in Julia set for \( Q_{-3/4} \). The width of the frame is about 3.5, and it is centered at the origin. (Notice the similarity with Figure 13.) Verify that the 2-cycle coincides with the indifferent fixed point for this system. Verify that the \( c \)-value for this example is on the cardioid drawn on page 243. Notice that this Julia set has an infinite number of conspicuous points, each of which is indexed twice. Can you identify these points and their indices?

6. Whenever zero is part of a cycle, the cycle is attracting. Explain why.

7. Refer to item 3 on page 27 and add the curve of indifferent 2-cycles to the diagram that appears on page 243.

8. You may have been wondering whether it is possible for \( Q_c \) to have more than one finite attractor. The answer is no, as Julia proved in his Critical Orbit Theorem:

   In order for a rational function \( F \) to have an attracting cycle, that cycle must attract the orbit of at least one critical point.

To say that a function is rational simply means that it is a quotient of two polynomials. A critical point of \( F \) is a solution to \( F'(z) = 0 \). In the light of this result — for which no proof is forthcoming — re-examine the examples you have seen so far.

9. Applying \( Q_c \) repeatedly to zero generates the critical orbit

   \[
   0, \ c, \ c^2 + c, \ (c^2 + c)^2 + c, \ ((c^2 + c)^2 + c)^2 + c, \ldots ,
   \]

   which can behave in a variety of ways. It can approach infinity, it can approach a finite cycle, it can be a finite cycle, it can be an eventual cycle, or — none of the above — it can wander aimlessly in the complex plane. Classify the following: \( c = -2, \ c = -\frac{3}{4}, \ c = -1, \ c = -1 + i, \) and \( c = i \).
Dynamic Fractals

The Mandelbrot set consists of all those \( c \)-values for which the critical orbit
\[ 0 , c , c^2 + c , (c^2 + c)^2 + c , \ldots \]
does not approach \( \infty \).

1. It is not difficult to give a complete description of the real values of \( c \) that belong to the Mandelbrot set. First, use graphical analysis (i.e., web diagrams — examples appear on pages 212, 213, and 220) to show that every \( c \)-value between \(-2\) and \( \frac{1}{4} \), inclusive, is a Mandelbrot point. You should be able to express your graphical reasoning in terms of inequalities. Finally, show that all the other real \( c \)-values allow the critical orbit to escape to infinity.

2. Suppose that \( 2 < |c| \). Show that the critical orbit approaches \( \infty \), thus proving that the entire Mandelbrot set is contained in the disk \( |c| \leq 2 \). (See item 4 on page 27.)

3. When \( c \) is not in the Mandelbrot set, then the Julia set of \( Q_c \) no longer encloses the origin, for the (critical) orbit of zero escapes to \( \infty \). The Julia set is in fact disconnected. Explain why it still includes the same diverse array of repelling cyclic points, however.

4. Figure 22 shows a very small neighborhood of the left-hand fixed point for \( Q_{-1} \), whose filled-in Julia set appears in Figure 13. With the help of a ruler, comment on the sizes and placements of the visible black components of the filled-in Julia set. How does the derivative enable you to predict what you see?

5. Among all the pictures of Julia sets you have seen so far, item 4 on page 25 stands out, for it does not show a filled-in set. The only points actually plotted are Julia points. This picture was produced by the Inverse-Image Method, abbreviated from now on as IIM. Here is the recipe: Take any member of the Julia set (a repelling fixed point, for example), calculate its immediate ancestors, and plot them. Now randomly choose one of these ancestors, calculate its immediate ancestors, and plot them. Repeat this procedure until the picture fills in. The IIM seems fast and simple. Why does it work? In particular, why do you not have to worry about small computational errors during the process? What would happen if the word random were left out of the instructions? Could the IIM be applied to a Newton-Raphson example?

6. When drawing filled-in Julia sets, points are colored according to how long it takes for their orbits to cross the escape threshold — a suitably large circle. The black points represent orbits that never cross this threshold. It is thus necessary to set an upper limit on how many iterations will be tried. What if it is set too low? What if it is set too high?

The drawing method used in almost all the examples so far might be called the Center-Only Method (abbreviated COM), because the color of the pixel is determined by the dynamic behavior of its center — other points of the pixel do not influence the decision.
Dynamic Fractals

The Mandelbrot set is shown in Figure 25. Its two largest components are introduced as simple geometric curves in item 7 on page 247. The cardioid encloses all the $c$-values for which $Q_c$ has an attracting 1-cycle. The circle encloses all the $c$-values for which $Q_c$ has an attracting 2-cycle. The cardioid and circle themselves consist of $c$-values for which $Q_c$ has an indifferent cycle. Their point of tangency, $c = -\frac{3}{4}$, represents the example (item 5 on page 246) in which the 2-cycle has merged with an indifferent 1-cycle.

1. Another familiar cardioid point is $c = \frac{1}{4} + \frac{1}{2}i$. This indifferent example appears first on page 28. Notice that there is a small disk attached tangentially at $c$ to the cardioid, directly above the cusp of the cardioid. Can you guess what the $c$-values inside this disk have in common? Can you also guess what is special about the center of this disk?

2. Verify that the highest point of the cardioid is the $c$-value that corresponds to an indifferent 3-cycle (item 3 on page 246). Notice that there is a small disk attached tangentially at this $c$-value. Can you guess what the $c$-values inside this disk have in common? Can you also guess what is special about the center of this disk? Look on page 244 to find its coordinate.

3. Given that the segment $-2 \leq x \leq \frac{1}{4}$ of the real axis is known to be part of the Mandelbrot set (see item 1 on page 32), it may surprise you not to see it in Figure 25. It does not appear because lines are thin, and thus do not show up well in high-resolution pictures. (You will soon learn ways to make the line conspicuous, however.) As Mandelbrot did when he looked at his first computer-generated pictures, you may notice several specks in the picture, which would pass for random dirt, were it not for their symmetric placement. It would be difficult not to be curious about them. To whet your curiosity, consider the value $c_1$ discovered in item 2 on page 244. It should be somewhere in the Mandelbrot set. Why is this, and where is it found in Figure 25?

4. Figure 26 shows the filled-in Julia set for the indifferent example $Q_{1/4}$. Make enough calculations to identify some of the conspicuous features. In particular, what are the dimensions of this set? What can be said about the field lines as they approach the Julia set?

5. For how many $c$-values does $Q_c$ have a superattracting 4-cycle? For how many $c$-values does $Q_c$ have a superattracting 5-cycle?
Dynamic Fractals

Superattracting cycles.

The equation \( Q^n_c(0) = 0 \) can be solved — in theory, anyway — to find the centers of all the \( n \)-cycle components of the Mandelbrot set. The cases \( n = 1 \), \( n = 2 \), and \( n = 3 \) have already been considered. Here are some more examples. First the 4-cycle centers:

\[
\begin{align*}
c_1 &= -1.9407998065294848 \\
c_2 &= -1.3107026413368329 \\
c_3 &= -0.1565201668337551 + 1.0322471089228318i \\
c_4 &= -0.1565201668337551 - 1.0322471089228318i \\
c_5 &= 0.2822713907669139 + 0.5300606175785253i \\
c_6 &= 0.2822713907669139 - 0.5300606175785253i \\
\end{align*}
\]

Next, the 5-cycle centers:

\[
\begin{align*}
c_1 &= -1.9854242530542053 \\
c_2 &= -1.8607825222048549 \\
c_3 &= -1.6254137251233037 \\
c_4 &= -1.2563679300681808 + 0.3803209634727225i \\
c_5 &= -1.2563679300681808 - 0.3803209634727225i \\
c_6 &= -0.5043401754462440 + 0.5627657614529820i \\
c_7 &= -0.5043401754462440 - 0.5627657614529820i \\
c_8 &= -0.1980420993642538 + 1.1002695372926985i \\
c_9 &= -0.1980420993642538 - 1.1002695372926985i \\
c_{10} &= -0.0442123577040706 + 0.9865809762808928i \\
c_{11} &= -0.0442123577040706 - 0.9865809762808928i \\
c_{12} &= 0.3592592247580074 + 0.6425137371385423i \\
c_{13} &= 0.3592592247580074 - 0.6425137371385423i \\
c_{14} &= 0.3795135880159237 + 0.3349323055974976i \\
c_{15} &= 0.3795135880159237 - 0.3349323055974976i \\
\end{align*}
\]

Because the equation \( Q^n_c(0) = 0 \) is real, its non-real solutions occur in conjugate pairs. Among the \( 2^n-1 \) solutions, there are always some redundant ones (for example, the 1-cycle component always appears), and these have been removed from the above lists.

Notice that most of these \( c \)-values are not close enough to the main cardioid to be centers of components that are \textit{attached} to it. In fact, only six of the above twenty-one entries represent components that are \textit{attached} to the cardioid — which ones?
Dynamic Fractals

Superattracting 6-cycles.

Here are the $c$-values for which $Q_c$ has a superattracting 6-cycle:

\[
\begin{align*}
c_1 &= -1.9963761377111938 \\
c_2 &= -1.967732163929287 \\
c_3 &= -1.9072800910653020 \\
c_4 &= -1.7728929033816238 \\
c_5 &= -1.4760146427284299 \\
c_6 &= -1.2840849255256857 + 0.4272688960406860i \\
c_7 &= -1.2840849255256857 - 0.4272688960406860i \\
c_8 &= -1.1380006666509645 + 0.2403324012620983i \\
c_9 &= -1.1380006666509645 - 0.2403324012620983i \\
c_{10} &= -0.5968916446451270 + 0.6629807445770296i \\
c_{11} &= -0.5968916446451270 - 0.6629807445770296i \\
c_{12} &= -0.2175267470305110 + 1.1144542658732927i \\
c_{13} &= -0.2175267470305110 - 1.1144542658732927i \\
c_{14} &= -0.1635982615520226 + 1.0977806428882722i \\
c_{15} &= -0.1635982615520226 - 1.0977806428882722i \\
c_{16} &= -0.1134186559494366 + 0.8605694725015731i \\
c_{17} &= -0.1134186559494366 - 0.8605694725015731i \\
c_{18} &= -0.0155703860209023 + 1.0204973664982890i \\
c_{19} &= -0.0155703860209023 - 1.0204973664982890i \\
c_{20} &= 0.3598927390125790 + 0.6847620202118129i \\
c_{21} &= 0.3598927390125790 - 0.6847620202118129i \\
c_{22} &= 0.3890068405697712 + 0.2158506508708191i \\
c_{23} &= 0.3890068405697712 - 0.2158506508708191i \\
c_{24} &= 0.3965345700324150 + 0.6041818104889888i \\
c_{25} &= 0.3965345700324150 - 0.6041818104889888i \\
c_{26} &= 0.4433256333996235 + 0.3729624166628465i \\
c_{27} &= 0.4433256333996235 - 0.3729624166628465i \\
\end{align*}
\]

Two of these $c$-values represent disks attached to the main cardioid, three are period-doubling disks that bud from the three 3-cycle components, two are period-tripling disks that bud from the 2-cycle disk, and twenty are satellite components, four of which are skewered on the real axis.
Dynamic Fractals

1. If a list of 7-cycle centers were made, as was just done for the 6-cycle centers, how long would the list be, and how would the centers be arranged in groups?

2. If a list of 8-cycle centers were made, as was just done for the 7-cycle centers, how long would the list be, and how would the centers be arranged in groups?

3. Figures 32, 33, and 34 show a trio of Julia sets that are all extracted from the same 5-cycle component, a disk attached to the main cardioid. The corresponding \( c \)-values are \(-0.494 + 0.583i\), \(-0.50434 + 0.56277i\), and \(-0.514 + 0.543i\). Figure 33 shows the superattractor. Compare and contrast these examples.

4. Explain why most rational functions have 3-cycles. In order for such a 3-cycle to be superattracting, what must be true?

5. Find all cycles and classify them for (a) \( F(z) = \frac{z - \frac{1}{z}}{z} \); (b) \( G(z) = \frac{z - \frac{1}{z}}{z + 1} \).

6. Figure 35 shows the filled-in Julia set for \( c = 0.28989123 + 0.45454003i \). This is the point on the Mandelbrot cardioid that results when \( \theta = (-2 + \sqrt{5})360 \) is substituted into the cardioid equations on page 243. This example, based on an irrational value of \( \theta \), differs in significant respects from all the other indifferent examples seen so far. How?

7. The figure at right is a web diagram. It shows what happens when a real-valued function is applied recursively to a seed value. Let \( Q_c(x) = x^2 + c \), and let \( p \) be the right-hand fixed point. For \(-2 \leq c \leq \frac{1}{4}\), show that the web diagram for the orbit of zero stays inside the square whose vertices are \((p, p), (-p, p), (-p, -p), \) and \((p, -p)\).

8. Suppose that \( F(p) = q \), \( F(q) = r \), and \( F(r) = p \), and let \( G = F^3 \). Apply the Chain Rule to show that \( G'(p) \), \( G'(q) \), and \( G'(r) \) all have the same value. What is the significance of this common value?

9. When \( c = \frac{1}{4} \), the indifferent fixed point \( p = \frac{1}{2} \) is the limit of the critical orbit

\[
0, \quad \frac{1}{4}, \quad \frac{5}{16}, \quad \frac{89}{256}, \quad \frac{24305}{65536}, \quad \ldots
\]

Draw a web diagram to illustrate this, then explain why it would still not be correct to refer to \( p \) as an attracting fixed point.
Dynamic Fractals

1. When \( c = -0.75 + 0.0001i \), it takes 31416 applications of \( Q_c \) for the critical orbit to jump outside the circle \( |z| = 2 \). What does this say about the accuracy of the Julia and Mandelbrot diagrams you have seen?

2. Recall that \( c = \frac{1}{2}\text{cis} \theta - \frac{1}{4}\text{cis} 2\theta \) parametrizes the Mandelbrot cardioid, and that — for such \( c \)-values — the fixed point for \( Q_c \) is \( p = \frac{1}{2}\text{cis} \theta \), with \( Q'_c(p) = \text{cis} \theta \). The \( c \)-values that result from taking \( \theta = 360m/n \) (degrees) serve as gateways into the \( n \)-cycle components that are attached tangentially to the cardioid. In other words, as \( c \) moves across the cardioid at such a point of tangency, the left-hand fixed point of \( Q_c \) experiences an \( n \)-fold pinch from the Julia set (as do all the ancestors of this fixed point), and an \( n \)-cycle becomes the finite attractor. In all, the Mandelbrot set has 120 small 8-cycle components; how many of them are attached to the cardioid?

3. A set \( S \) is called closed if sequences in \( S \) converge only to limits that are in \( S \). For example, the real interval \( 0 \leq x \leq \pi \) is a closed set, whereas neither the set of all rationals nor the interval \( 0 < x < 2 \) is closed. What about the interior of the unit circle? What about the set of numbers of the form \( 1/n \), where \( n \) is a positive integer? What about the orbit of zero for the function \( Q_{1/4} \)? What about the ancestral tree of the repelling fixed point of \( Q_{-5/16} \)?

4. By adding any missing sequential limits, you form the closure of a set.
(a) What is the closure of the interior of the unit circle?
(b) What is the closure of the set of all points \((x, y)\) with rational coordinates?
(c) What is the closure of the ancestral tree of 1 in the Julia set for \( Q_0 \)?
(d) Is it possible for two different sets to have the same closure?
(e) What is the closure of the graph of \( y = \sin(1/x) \)?

5. The precise definition of the Julia set of a rational function is the closure of the set of repelling periodic points. Explain why this definition is needed to describe accurately examples such as those displayed in Figures 14, 20, 26, and 35.
Critical orbits for $Q_c$ start at the origin. If there is a finite attractor, the critical orbit will approach it. For example, Figure 36 shows the critical orbit that corresponds to $c = 0.12 + 0.59i$. Figures 37 and 38 do the same for $c = 0.12 + 0.58i$ and $c = 0.105 + 0.58i$, respectively. The pinwheel aspect of these orbits suggests that the examples were chosen in the vicinity of the budding point of a 7-cycle component, and in fact the indifferent example $c = 0.113981749997 + 0.595934890870i$ is nearby.

Figure 39 shows the first one hundred thousand terms of the critical orbit. The slow convergence of this sequence is not surprising, because of the indifferent fixed point. The budding point $c$ on the cardioid corresponds to $\theta = \frac{2}{9}(360)$. This also tells you where to find $c$ (which is the term that follows zero in the critical orbit) in all four examples. It is at the end of spoke number 2, counting counterclockwise from the origin, which is at the end of spoke number 0.

As a striking contrast, Figure 40 shows five different orbits for the indifferent example $c = 0.1265612308279 + 0.5896660980827i$, which is obtained by substituting $\theta = 72\sqrt{2}$ into the cardioid equation. This is an irrational angle, which means that $c$ is not a budding point for an $n$-cycle component. The most conspicuous feature of this example is that none of the orbits approaches the indifferent fixed point. Indeed, the critical orbit (the outermost one) lies in the Julia set, for that is where the origin is, having just been pinched. All the orbits for this quadratic function lead to invariant closed curves that surround the fixed point. This is a consequence of $Q_c'(p) = \text{cis}\, \theta$ being an irrational rotation.

It should also be noticed that these invariant curves are not completely filled by a single orbit, and that they are not generated in a continuous fashion. This would be evident if you saw them being drawn one dot at a time. An analogous example would be to plot the sequence $z_n = \text{cis}\, n\sqrt{2}$. The terms of this sequence hop discontinuously around the unit circle, and — after thousands have been plotted — only appear to fill it.
Dynamic Fractals

1. Figure 19 shows a superattracting 3-cycle Julia set. Identify the three field lines that land at the left-hand fixed point. Then identify the three field lines that land at its immediate ancestor. Do you think that these lines have to spiral in order to land?

2. As \( c \) drifts across a component of the Mandelbrot set, the accompanying Julia set evolves, from its fat, indifferent appearance (when \( c \) enters the component), to a leaner appearance (when \( c \) is near the center of the component), to a fat, indifferent appearance again, the result of new pinches. Contrast Figures 19 and 21 with Figure 41 (and the sevenfold magnification in Figure 42), which shows an indifferent 3-cycle just as a 6-cycle pinches it. This is called period-doubling. Identify the field lines that enter the two pinch points closest to the origin. Notice how this filled-in Julia set is a blend of the examples shown in Figures 19 and 20. In the Mandelbrot diagram, find the \( c \)-value that produced this example (do not however try to compute its value).

3. In the preceding example, a 3-cycle has re-entered the Julia set. While \( c \) was traveling across the upper 3-cycle component of the Mandelbrot set, the 3-cycle was attracting, but now a 6-cycle is about to take its place among the black points. Notice that each part of the 3-cycle has acquired two new indices. What are they? The old 3-cycle indices — \( \frac{1}{7}, \frac{2}{7}, \) and \( \frac{4}{7} \) — still mark the left-hand fixed point (which is where the 3-cycle first left the Julia set).

4. Other quadratic functions: Why have they not been given the same attention as the examples \( Q_c \) have received? The simple reason is that you would see the same Julia sets! For example, the quadratic function \( F(z) = 3z - z^2 \) produces a system that is isomorphic to the system defined by \( Q_{-3/4} \). Make calculations to confirm this.

5. Figure 61 shows the filled-in Julia set for \( c = 0.31847226665803 + 0.04125736992217i \). This is the superattracting representative of an \( n \)-cycle component of the Mandelbrot set. Do a little detective work and figure out what \( n \) is, and which component the example is taken from.

The difficult irrationally indifferent examples considered on pages 36 and 38 can be understood by thinking of them as rational, but with a very large denominator. The superattracting example in Figure 61 already shows how complicated it is for a fixed point to be pinched by an 11-cycle. It is hard to imagine all the detail, given that every ancestor of the fixed point is pinched in the same way! Moreover, in the indifferent case that serves as the gateway to this component, the fine detail would be even more crowded.

Figure 65 shows the filled-in Julia set for \( c = 0.2803470120164 + 0.4684448909879i \), which is a slightly more difficult example. This \( c \)-value is chosen from the \( \frac{6}{25} \) bud on the main cardioid. Notice how the 25 limbs that are attached to the left-hand fixed point are barely distinguishable. Notice also that this filled-in Julia set looks a lot like the irrational case in Figure 35. The resemblance would have been even stronger if Figure 65 had been chosen from a 1000000-cycle component.
**Dynamic Fractals**

The period-doubling route to chaos is the name given to the examples $Q_c$ that result from choosing $c$ on the real axis, starting at 0 and decreasing to the chaos point whose coordinate is $c = -1.401155\ldots$. Except for $c$-values that produce indifferent examples, the critical orbit of $Q_c$ is attracted to cycles of lengths 1, 2, 4, 8, \ldots

For $c$ between $-0.75$ and 0, the critical orbit is attracted to a fixed point. Initially ($c = 0$) the Julia set is a circle, but it immmEDIATELY becomes infinitely wrinkly (the case $c = -0.3125$ shown in Figure 6 is typical), and deepening valleys finally result in the simultaneous pinching seen in the indifferent case $c = -0.75$ shown in Figure 20.

As $c$ continues to decrease, the critical orbit is suddenly attracted to a 2-cycle, which became real when the pinch occurred. This phenomenon — an attracting fixed point suddenly splitting into an attracting 2-cycle — is known as a bifurcation. The pinches become more pronounced. The superattracting case $c = -1$ is shown in Figures 13 and 17. One member of the attracting 2-cycle is found in the largest filled-in component (the component that contains 0), and the other is found in the component immediately to its left. As $c$ decreases toward $-1.25$, a new system of valleys forms, preparing to pinch the 2-cycle and its ancestors. Meanwhile, the two members of the 2-cycle continue to drift apart. Figure 43 shows the weakly attracting case $c = -1.2$, and Figure 44 shows the indifferent case $c = -1.25$. The 2-cycle has been pinched, as have all its ancestors. The 2-cycle itself is now back in the Julia set — where it started. One member is at the pinch point just to the right of 0, while the other is at the pinch point just to the left of the next-to-largest component to the left of 0.

As $c$ continues to decrease, the critical orbit is suddenly attracted to a 4-cycle, which became real when the most recent pinches occurred. This is another bifurcation, whereby each of the elements of the attracting 2-cycle splits in two, creating an attracting 4-cycle. The 4-cycle resides in each of four filled-in components — the components adjacent to the pinches described above. The superattracting case $c = -1.3107\ldots$ is shown in Figure 45. As $c$ continues to decrease, this attracting 4-cycle will of course undergo a bifurcation, suddenly splitting into an attracting 8-cycle \ldots

This bifurcation phenomenon occurs more and more rapidly as $c$ decreases. This corresponds to the rapidly decreasing sizes of the disk-like Mandelbrot components that occur on the negative real axis. The limit of the centers of these $2^n$-cycle components (approximately $-1.401155$) is an irrational $c$-value that defines a new type of Julia set. The function $Q_c$ has neither an attracting nor an indifferent cycle, and the infinite sequence of pinchings has reduced the filled-in components to nothing. The new Julia set is shown in Figure 46. Consisting exclusively of branches, it encloses nothing. Such Julia sets are called dendrites. The critical orbit, which of course is found in the Julia set, behaves chaotically. It does not escape to infinity, approach a finite cycle, or become one. Instead, this is a case of a critical orbit wandering aimlessly in the complex plane, alluded to in item 9 on page 32. (In this case, the orbit actually wanders aimlessly in a segment on the real axis.)
Dynamic Fractals

Other routes to chaos.

One can discover new Julia sets by means of period-tripling. For example, try to imagine the result of period-tripling the rabbit shown in Figure 19. Then look at Figure 47, which shows one of the possibilities. It is a Julia set that encloses a superattracting 9-cycle. Its $c$-value is $-0.031552974812924 + 0.7907831754064002i$, which is the center of the right-hand period-tripling bud attached to the upper 3-cycle bud attached to the main cardioid. Notice how the rabbit pattern has been inserted into itself. As complicated as this example is, it is mild compared to what you would see if you were to enter a long sequence of period-tripling buds. This can in fact be done forever. Such a sequence of central $c$-values would approach a limiting $c$-value, which would belong to the Mandelbrot set because the Mandelbrot set is closed. The Julia set corresponding to this $c$-value would be a most intricate fractal, called a dendrite because all the black points have been pinched away. In other words, the Julia set, although connected, would be all branches and surround nothing.

Thus there are many routes to chaos, which are in fact suggested by Figure 25. You have probably already noticed that every component of the Mandelbrot set is ringed by an infinite assortment of tangentially attached components. These are actually arranged in a simple parametric fashion. The following description is easily visualized in terms of the 2-cycle disk (which happens to be a real circle), but it applies with minor changes to any component $C$:

The components that are tangent to $C$ can be indexed counterclockwise by an angular parameter $\theta$ (interpreted in degrees), so that the component from which $C$ buds is assigned the value $\theta = 0$, the period-doubling component is assigned the value $\theta = 180$, the first period-quadrupling component is assigned the value $\theta = 90$, the second period-quadrupling component is assigned the value $\theta = 270$, and so on — in general, there is a bud corresponding to any parameter value of the form $360m/n$. Entering this bud corresponds to an $n$-fold pinch. Put another way, the attracting cycle associated with $C$ splits into a cycle with $n$ times as many terms, a process that is called bifurcation when $\theta = 180$, but which could perhaps be called $n$-furcation in general. For example, the Julia set shown in Figure 47 has undergone two 3-furcations — specifically, $\theta = 120$ and $\theta = 120$, while the Julia set in Figure 41 has undergone a 3-furcation and then a bifurcation — specifically, $\theta = 120$ and then $\theta = 180$.

Cardioid-like components are special because they do not bud from other components. With the exception of the main cardioid, the cusp marks the end of a filament that connects the component to the rest of the Mandelbrot set. Filaments begin at chaos points (as described on page 40) where an infinite sequence of period-multiplications has led to a chaotic critical orbit. See the next page for more examples.
Dynamic Fractals

There are many period-multiplying routes to chaos. You have already begun to explore the one that is described by the infinite sequence of angles 120, 120, 120, 120, \ldots, which defines a period-tripling scenario. Figure 47 shows the Julia set that results after two steps of this process, and Figures 48 and 49 show the next two steps. Each of these Julia sets encloses a superattracting cycle. The periods are 9, 27, and 81, respectively. There being very little left to be pinched, you can regard the 81-cycle example as a good approximation to what lies at the end of this sequence of period-tripling steps, which is leading you toward the chaos point $c = -0.023641169 + 0.783660651i$. The actual dendrite that this $c$-value defines would of course be invisible if the COM were used to draw it.

There are countless ways to reach chaos points, for there are infinitely many period-multiplying exits from each component of the Mandelbrot set, thus infinitely many choices for each step of the process. Because two different paths can not lead to the same chaos point, this means that there is a staggering variety of ways to leave the main part of the Mandelbrot set, without actually leaving the Mandelbrot set (i.e., without disconnecting the Julia set).

When the sequence of angles is constant (as above, and as in the period-doubling example described on page 40 and shown in Figure 46), it is conjectured that the sequence of centers approaches its limit in an essentially geometric fashion. In other words, successive center-to-center vectors have nearly a fixed ratio, the ratio approaching a constant as the sequence converges.
Dynamic Fractals

**Satellite Mandelbrot sets** account for those specks of dust that were noted in item 3 on page 34. Infinitely numerous and *very* tiny, they are needed to account for most of the superattracting centers that the equations $Q^n_c(0) = 0$ promise (see the lists on pages 35 and 36, for example). The Julia sets that correspond to $c$-values found in these subsets of the Mandelbrot set are especially interesting, for they are also built from miniatures of what you have already seen.

Figure 18 shows the Julia set that is the superattracting representative of the cardioid-like component of the biggest satellite. The appearance of this filled-in Julia set suggests that infinitely many copies of a filled-in circle have been inserted into what would otherwise be a (virtually invisible) dendrite. The insertion points correspond to the ancestral tree of 0. The dendritic skeleton of this Julia set is a consequence of pushing $c$ along a filament to get it to where it now is; the filled-in circle represents what you would see if the $c$-value were actually chosen from the corresponding place in the main cardioid.

Another illustration of this phenomenon is shown in Figures 50 and 51. The $c$-value for this Julia set is $-1.757783060083 + 0.0137961433695i$, which is the center of the upper period-tripling component of the same satellite. This value would appear in a list of superattracting 9-cycles, for the base period of this satellite is 3. Figure 51 is a magnification of the central portion of Figure 50. The attracting 9-cycle is housed in three of the rabbits, two of which are barely visible in Figure 50. The middle one must of course contribute to the cycle, for that is where 0 is found. The second rabbit shows only as the leftmost tick on the left part of the dendrite, and final one appears as the largest tick on the right part of the dendrite. In the first and third rabbits, the attracting orbit is found in the torso and the *legs* (rather than the torso and the ears). Both figures were drawn by using the filament-revealing DEM.

As $c$ wanders through this period-3 satellite, the corresponding Julia sets thus appear as infinitely many synchronized miniatures, linked together by the dendritic frame — much like watching several television sets that are all tuned to the same channel. By the way, this pastime of tracing a path of $c$-values somewhere in the Mandelbrot set, while watching the resulting display of Julia sets has been aptly named the *Julia Promenade*.

The most astonishing (unproved) conjecture about these satellites is that they are *densely threaded on every filament*! In particular, given any segment of the (straight) real axis filament, no matter how tiny, there are infinitely many of these satellites to be found there. Most are too tiny to be seen, but all are complete in every detail.

Figure 62 shows a satellite of the Mandelbrot set. Its cardioid-like component represents attracting 5-cycles, and it is centered at $c = -1.6254137251233037 \ldots$ on the real axis. Figure 63 shows the filled-in Julia set for this $c$-value, and Figure 64 magnifies the part of Figure 63 that contains 0.
Dynamic Fractals

Misiurewicz points are among the most conspicuous points in the Mandelbrot set. These are the $c$-values where filaments branch and end. The corresponding Julia sets are dendritic. What makes these examples especially interesting is that they are in principle computable, for they occur precisely when the critical orbit

$$0, c, c^2 + c, (c^2 + c)^2 + c, ((c^2 + c)^2 + c)^2 + c, \ldots$$

is not cyclic, but is eventually cyclic. This accounts for one of the intriguing possibilities mentioned in item 9 on page 32.

The two simplest examples are $c = -2$ (zero is eventually fixed) and $c = i$ (the critical orbit terminates in a 2-cycle). Both of these $c$-values occur at the ends of filaments. The Julia set of $Q_i$ is shown in Figure 52. The Julia set of $Q_{-2}$ is shown below. Consisting solely of the segment from $-2$ to 2, it is one of those rare (non-fractal) Julia sets that can be drawn by hand.

1. In addition to $-2$, there are infinitely many $c$-values for which the critical orbit of $Q_c$ is eventually fixed. Three of them can be found as the roots of a certain cubic equation. Find this equation, and — if you have the resources — solve it numerically. The single real solution is a branch point that appears near the middle of the main filament, while the conjugate complex solutions are the ends of conspicuous filaments that appear above and below the 3-cycle disks.

2. One of the figures below shows the graph of $P(x) = x^2 - x - 2$ and the line $y = x$; the other shows the graph of $R(x) = x^2 + x - 3$ and the line $y = x$. Examine these two figures and then make calculations that show why the Julia sets of $P$ and $R$ are congruent.

3. Given any polynomial $P$ of degree $n$, there is a polynomial of the same degree — with coefficient zero on the term of degree $n - 1$ — whose Julia set is congruent to the Julia set of $P$. Explain why.
Dynamic Fractals

1. One of the figures below shows the graph of \( P(x) = x^2 - x - 2 \) and the line \( y = x \); the other shows the graph of \( S(x) = 2x^2 - x - 1 \) and the line \( y = x \). Examine these two figures and then make calculations that show why the Julia sets of \( P \) and \( S \) are similar.

\[ \begin{align*}
\text{Graph of } P(x) & = x^2 - x - 2 \\
\text{Graph of } S(x) & = 2x^2 - x - 1
\end{align*} \]

2. Given any polynomial \( P \) of degree \( n \), there is a polynomial of the same degree — with leading coefficient 1 — whose Julia set is similar to the Julia set of \( P \). Explain why.

3. In making up a catalogue of cubic Julia sets, it is sufficient to consider only examples of the form \( C(z) = z^3 + mz + b \). Why?

4. Invent a cubic polynomial that has two superattracting finite fixed points. Make a rough sketch of what you think the Julia set of your polynomial would look like. Remember that infinity is an attracting fixed point.

5. Draw the graph of the real equations \( y = (x^2 - 1)^2 - 1 \) and \( y = x \) on the same coordinate axes. You should see four points of intersection. Identify them and describe their significance.

6. To draw the Julia set for \( Q_c(z) = z^2 + c \), the IIM repeatedly applies the functions \( F(z) = \sqrt{z - c} \) and \( G(z) = -\sqrt{z - c} \) to a seed point, the function selection always being random. What would happen if instead the functions were selected \( F, G, F, G, \ldots \) in a strictly alternating pattern?

7. Describe the effect of applying the linear function \( L(z) = 3 + \frac{1}{2}i(z - 3) \).

8. Suppose that \( F \) and \( G \) represent isomorphic dynamic systems, which means that there is a differentiable function \( \phi \) with the property that \( F(\phi(z)) = \phi(G(z)) \) holds for all relevant values of \( z \). Let \( p \) be a fixed point for \( G \). Show that \( \phi(p) \) is a fixed point for \( F \), and that these fixed points have the same multiplier.

9. It so happens that \( Q_c \) has an attracting 5-cycle when \( c = -1.6254137 \) and also when \( c = -1.8607825 \). What about the values of \( c \) in between?
Dynamic Fractals

1. Consider the real graph \( y = Q_c^3(x) = Q_c(Q_c(Q_c(x))) \) for various values of \( c \). The cases \( c = -1, c = -1.3, \) and \( c = -1.76 \) are shown from left to right below, along with the graph of \( y = x \).

Notice that the third example shows eight intersections (the maximum possible) with the line \( y = x \). What is the significance of these intersections? Does the polynomial graph cross the line with eight different slopes?

In case you are skeptical about the intersections in the lower left corner of the third figure, a hundredfold magnification is shown at right. Not only are two intersections in clear view, but this figure looks very familiar. Where have you seen it before?

2. Assume that \( c \) is a real number less than \( \frac{1}{4} \) and that \( n \) is some positive integer. If the real graphs \( y = Q_c^n(x) \) and \( y = x \) are drawn on the same coordinate axes, then the line will intersect the polynomial graph in at least two places. Why?

3. If the graphs of \( y = x \) and \( y = Q_c^5(x) \) intersect in more than two places, then it is possible to find five intersections in the figure, at all of which the polynomial slopes are the same. Explain why this is so.

4. To draw the Julia set for \( Q_c(z) = z^2 + c \), the IIM repeatedly and randomly applies the functions \( F(z) = \sqrt{z - c} \) and \( G(z) = -\sqrt{z - c} \) to a seed point. These functions are of course closely related (being the two branches of the inverse to \( Q_c \)). What would happen if two arbitrary functions were selected to play the role of \( F \) and \( G \)? For example, consider the simple linear functions

\[
F(z) = 1 + \frac{2}{3} i(z - 1) \quad \text{and} \quad G(z) = -1 + \frac{2}{3} i(z + 1).
\]

Would there be any harm done if you were to try running IIM on these functions, even though this seems to have nothing whatever to do with drawing a Julia set?
Dynamic Fractals

Other quadratics provide no new Julia sets.

1. Show that the system \( F(w) = 3w - 3w^2 \) is isomorphic to the system defined by \( Q_{-3/4} \), and that the two Julia sets are similar, in the strict geometric sense. Show that, in fact, there is a linear change of variables \( w = mz + b \) that makes \( F(mz + b) = mQ_{-3/4}(z) + b \) for every \( z \), by finding the \( m \) and \( b \).

2. On the basis of your calculations in the preceding item, sketch the Julia set for \( F \).

3. Show that any quadratic system \( F(w) = aw^2 + kw + d \) is isomorphic to a system \( Q_c \), by means of a linear change of variables \( w = mz + b \). The values of \( c, m, \) and \( b \) of course depend on the coefficients \( a, k, \) and \( d \) of the given example. They are determined by the requirement that \( F(mz + b) = mQ_c(z) + b \) hold for every \( z \). Find formulas for \( c, m, \) and \( b \), in terms of \( a, k, \) and \( d \). By the way, you should notice that \( a \neq 0 \) is necessary for your formulas to work.

4. The concept of isomorphic systems can perhaps be better understood by the following familiar analogy: To find the square roots of a complex number in Cartesian form \( x + yi \), it is usually easiest to first convert the problem to polar form, replacing \( x + yi \) by the equivalent \( r \text{cis} \theta \). Then, after performing the simple root-finding step \( \pm \sqrt{r} \text{cis} \frac{1}{2} \theta \), the original problem is solved by conversion back to Cartesian form.

Complete this analogy by finding a square-root process that is finite (no limits) and that is described exclusively in terms of Cartesian coordinates \( x \) and \( y \).

5. Is it possible for two quadratic systems of the form \( Q_c \) to be isomorphic to each other? If there is such an example, the change of variables might not be linear.

6. Find a quadratic polynomial that has \( 0 \rightarrow 1 \rightarrow i \rightarrow 0 \) as a 3-cycle. In your example, is this cycle attracting?
Dynamic Fractals

The Mandelbrot set is essentially a *catalogue* of Julia sets, with no dynamic structure of its own. In other words, every point in Figure 66 represents a *different* Julia example. You might therefore be wondering why *escape-time contours* appear in the diagram! The answer is simple: If two $c$-values happen to lie in the same exterior band, this means only that the two critical orbits — one for each example — took the same number of iterations to cross a common threshold circle on their way to infinity. Among other things, these escape-time contours provide a good way to see the invisible filaments and components that are an essential part of the connected Mandelbrot network.

**Seahorse Valley** is the narrow gap between the main cardioid and the 2-cycle disk attached to it. Its name is based on certain suggestive patterns, found in the Julia sets selected from this region of the Mandelbrot set, and found in the Mandelbrot set itself.

For example, to better appreciate the appearance of the Julia set shown in Figures 70, 71, and 72, begin by considering the $\frac{13}{29}$ component, shown in Figure 67, that buds from the main cardioid. It is the lower of the two large, disk-like components attached to the cardioid wall. Figure 68 shows the Julia set that encloses the corresponding superattracting 29-cycle.

The reason for the descriptive name *seahorse* has already become evident in the picture of the Julia set. It is also interesting that the conspicuous spiral points that are responsible for these characteristic seahorse tails are nothing more than the familiar (repelling) 2-cycle and its ancestral tree. To see why, just calculate the 2-cycle local multiplier for $c$-values in this region. The derivative multiplier (see item 3 on page 240) is approximately $4 + 4c = 4 + 4(-0.75 + 0.125i) = 1 + 0.5i = 1.12\text{cis}26.6$. This generates a spiral that expands by a factor of only about 4.5 for every 360 degrees of turning — a pleasant curve that can be appreciated with the unaided eye.

Now move the $c$-value out onto a filament that is connected to this 29-cycle component, then along the filament to a tiny satellite that is slightly above the center of Figure 67. A 2000-fold magnification of this satellite is shown in Figure 69. The base period of this satellite is 45, and its center is $c = -0.745428025196324 + 0.113009193279259i$.

During the move, the Julia set shown in Figure 68 is first pinched away to a lacy dendrite (by repeated period-multiplication), then tiny disks, rabbits, dragons — the complete Mandelbrot catalogue — are inserted at the ancestral tree of 0, when $c$ passes through a satellite.
Dynamic Fractals

The Julia set for \( c = -0.745428025196324 + 0.113009193279259i \) is difficult to draw. The Inverse-Image Method plots points accurately, but it does not uniformly reach all parts of the Julia set. To see this, compare the IIM-generated Figure 70 with the more accurate Figure 71. It is evident that only a small selection of Julia points appears in Figure 70. The conspicuous points are those that lie at the end of field lines that land without difficulty. Ancestors of the repelling fixed point (which correspond to dyadic field lines) dominate the view, but most of these ancestors still go undetected by the IIM, which misses almost all the detail.

In a figure drawn by the Center-Only Method, a pixel appears black if its central point belongs to the Julia set, or if this central point takes so long to cross the infinite threshold that the repetition counter reaches its maximal allowed value. For examples such as this one that are (nearly) dendritic, the COM therefore does not work well — most Julia points do not lie exactly at the centers of pixels, and are therefore invisible. In the current example, it is necessary to let the iteration counter run rather high to reach the “inner” parts of the configuration, but this in turn means that most pixels will be colored white. It is possible to compromise and deliberately lower the maximum iteration count, but this means that the figure will be dominated by “accidental” black pixels.

As Figure 71 shows, the Distance-Estimator Method is well-suited to this example. A pixel is colored black if it intersects the Julia set. This has the desirable effect of revealing filaments. This “thickening” can cause neighboring filaments to merge into an undifferentiated mess, however, which explains why satellite Mandelbrot sets usually have a furry appearance — the small ones are very “hairy.” To appreciate how intricate and lacy this Julia set is, examine the 100-fold magnification shown in Figure 72, which was also drawn using the DEM.

Because the \( c \)-value \(-0.745428025196324 + 0.113009193279259i\) is the center of a cardioid-like Mandelbrot component, it is not surprising to see in Figure 72 that the filled-in Julia set looks like a circular disk, in the vicinity of the origin (which is the central point).

Most of the black points in Figure 72 are attracted to the 45-cycle to which the origin belongs. There are also accidental black smudges mixed in with the true black pixels, of course. The ensemble reveals some familiar dynamic themes. In particular, notice the points that are ancestral to the left-hand fixed point (which belongs to the Julia set). These conspicuous points are surrounded by patterns that are reminiscent of a 29-equipped pinwheel (compare with the Julia set in Figure 68). The left-hand fixed point itself stands out in Figure 71. Notice also how the numerous seahorse tails spiral around points that are ancestral to the 2-cycle (which belongs to the Julia set). The 2-cycle itself stands out clearly in Figure 71.
Dynamic Fractals

1. What are the critical points of $F(z) = Q_c^2(z) = Q_c(Q_c(z))$? What are the critical points of $G(z) = Q_c^3(z) = Q_c(Q_c(Q_c(z)))$?

2. Consider the function $Q_{-1}$, whose filled-in Julia set is shown in Figure 13. Find ten members of the ancestral tree of 0, including at least two that are not real. Find these $z$-values in the picture.

3. Make up an example of a cubic polynomial that has both an attracting 2-cycle and an attracting 1-cycle (fixed point). Make a rough sketch of what you think the Julia set of your polynomial looks like. Your polynomial will probably have repelling fixed points as well. Find them and incorporate them into your sketch.

4. It is possible for a cubic polynomial to have two attracting 3-cycles. Is it possible for a cubic polynomial to have three attracting 2-cycles? Why?

5. Consider the function $C_k(z) = k z^2 (2 - z)$, where $k$ is a parameter.
   (a) Prove that $z = 0$ is a superattracting fixed point for every $C_k$.
   (b) What is the other critical point of $C_k$?
   (c) For what values of $k$ does $C_k$ have three real fixed points?
   (d) For what value of $k$ does $C_k$ have two superattracting fixed points?
   (e) For what values of $k$ does $C_k$ have an indifferent fixed point?
   (f) Find a value of $k$ for which $C_k$ has an attracting 2-cycle.

6. Find the linear function $L(z) = mz + b$ for which $L(2) = -1 + 2i$, $L(2i) = i - 2$, and $L(2 + 2i) = -2 + 2i$. What is the fixed point of $L$?

7. Suppose that $r$ is an arbitrary real number between 0 and 1, $\theta$ is an arbitrary angle measure between 0 and 360, and that $c$ is an arbitrary complex number. What is the effect of applying $G(z) = c + (z - c)rcis\theta$ repeatedly to an arbitrary seed value $z_0$?

8. Let $F(z) = \frac{1}{2} z$, $G(z) = 2 + \frac{1}{2}(z - 2)$, and $H(z) = 1 + 2i + \frac{1}{2}(z - 1 - 2i)$. Notice that all three functions are linear, with conspicuous fixed points. Let $z_0$ be one of the fixed points, and consider what happens when the system of functions $\{F, G, H\}$ is applied repeatedly and randomly to $z_0$. Although the computer will produce a quick graph of the resulting orbit, it is possible to apply some geometric insight instead.
Dynamic Fractals

The Newton-Raphson Method Revisited.

To solve an equation $E(z) = 0$, just set up the dynamic process $N(z) = z - \frac{E(z)}{E'(z)}$.

It is a remarkable fact that the roots to the given equation are thus transformed into superattracting fixed points for $N$:

1. Suppose that $E(p) = 0$. Prove that $N(p) = p$ and that $N'(p) = 0$.

2. Show that the only other critical points that $N$ can have are the solutions to $E''(z) = 0$.

3. Given a positive integer $k$ and a complex number $b$, set up the function $N$ to solve the equation $E(z) = z^k - b = 0$.

4. Set up the Newton-Raphson system for $z^3 + z - 2 = 0$.

The Newton-Raphson approach to solving equations seems almost trouble-free. Once the function $N$ has been set up, it is ready to lead you to all the roots of $E(z) = 0$, one at a time. In other words, almost any seed value you select will generate an orbit that converges (rapidly) to a root. As you have seen, there are some troublesome seed values. The Julia set, which is the closure of the set of all repelling periodic points, consists of troublesome seeds. The orbits that begin in the Julia set stay there, behaving chaotically.

5. A simple example of how a 2-cycle can be built into the Newton-Raphson method: Consider the equation $5z - z^3 = 0$. Set up the root-finding function $N$, then show that $z = 1$ is part of a 2-cycle for $N$. Find the immediate ancestors of this 2-cycle, and incorporate them in a sketch of the Julia set for $N$.

Even though there are indecisive points of all sorts built into the Newton-Raphson method, they still represent an insignificant part of the process, for they are (relatively) few in number, and — being part of the Julia set — they are repelling by nature, and thus invisible. Examples like the following therefore come as a surprise:

6. Consider the polynomial equation $4z^3 - (c^2 + 3)z + c^2 - 1 = 0$, where the parameter $c$ is $0.007 + 1.019i$. With the help of the computer, show that the origin is attracted to a 9-cycle when the Newton-Raphson method is applied. Moreover, the same is true for all seed values in a small neighborhood of the origin. In other words, there is a small disk surrounding the origin, all points of which are attracted by something other than a root of the equation! Notice that item 2 above alerts you to this possibility. How?

Joseph Raphson (1648-1715) was a contemporary of Newton.
The equation \(4z^3 - \left(c^2 + 3\right)z + c^2 - 1 = 0\) is solved by the dynamic system
\[
N(z) = \frac{8z^3 + 1 - c^2}{12z^2 - c^2 - 3}.
\]

The case \(c = 0.007 + 1.019i\) is shown in Figure 75. The three attractive basins defined by the roots are colored white and two shades of gray. Black is reserved for points that are attracted to a 9-cycle (instead of being attracted to a root of the polynomial). The image is centered at the origin, and is just large enough to include the roots \(z = 1\) and \(z = \frac{1}{2}(-1 \pm c)\).

Figure 76 shows a tenfold magnification of the center of Figure 75. The black points are all attracted to a 9-cycle, instead of to one of the polynomial roots. Three of the points of this cycle are found in the rabbit — two in the ears and one in the torso. The other six points of the cycle are found in the two rabbits that are barely visible left of center in Figure 75.

Why do you see rabbits here, in this non-quadratic Julia example? It is because a certain special value of \(c\) (close to \(0.007 + 1.019i\)) happens to make the graph of \(N^3\) tangent to the input-output graph (as shown on page 46). For those \(c\)-values that are suitably close to this special value, the resulting slight intersection creates a trap when \(N_c\) is applied repeatedly to the critical point 0, thereby opening the door to the entire Mandelbrot catalogue of cyclic behavior. See page 54 for a survey of these special \(c\)-values.

Figure 86 is a labeled version of Figure 75. One of the roots is the third-quadrant point \(\frac{1}{2}(-1 - c) = -0.4665 - 0.5095i\), and its unbounded attractive basin is colored light gray. This unbounded basin is an invariant set, isomorphic to the dynamic system defined by \(Q_0\) inside the unit circle. The root corresponds to the origin. Boundary points (all of which belong to the Julia set) correspond to points on the unit circle, and can be given angular labels that range counterclockwise from 0 to 1. Because \(\infty\) is a repelling fixed point for the root-finding process, the initial label 0 goes there. Its two immediate ancestors are the roots of \(12z^2 = c^2 + 3\). One of them is part of the border that separates light gray from white, and its label is therefore \(\frac{1}{2}\). (The other ancestor is the leftmost point in the dark gray region.) In general, each dyadic label in the light gray region corresponds to the nearest conspicuous 3-corner point, all of which are ancestral to \(\infty\). The points labeled \(\frac{3}{7}\), \(\frac{6}{7}\), and \(\frac{5}{7}\) form a 3-cycle, and the point labeled \(\frac{5}{14}\) is ancestral to this 3-cycle. Each 3-cycle point (which is a fixed point for \(N^3\)) appears as the “toe” of a rabbit’s foot.
Dynamic Fractals

Julia set quiz.

The IIIM was used to draw this rough outline of the Julia set of $Q_c$, for $c = -0.122 + 0.745i$. Upper-case letters have been used to mark some of the components of the basin of the attractive 3-cycle. Notice that the critical central component has been split into two halves, named $C_1$ and $C_2$. Lower-case letters have been used to mark the right-hand fixed point and some of its ancestors.

1. Except for the repelling fixed points, each part of this figure is mapped by $Q_c$ to another part of the figure. Give the details of this mapping, by telling the destination for each of the upper-case letters in the figure.

2. Several ancestors of the right-hand fixed point are indicated by lower-case letters in the figure. What are the indices of the corresponding field lines?

3. Identify the pinch point that is common to the components labeled $E$, $F$, and $G$. What is the index of the field line that lands at this point, after passing through the bay formed by components $E$ and $G$?
Dynamic Fractals

Parameter-Plane Charts.

The Mandelbrot set is a chart in the $c$-plane (the so-called parameter plane) that displays specific information about quadratic dynamic systems in the $z$-plane (the dynamic plane). That is, $c$ is in the Mandelbrot set if the function $Q_c$ generates a bounded orbit when it is applied to the critical point zero. An equivalent definition: $c$ is in the Mandelbrot set if the Julia set of $Q_c$ is connected.

The concept of a chart in the $c$-plane can be applied to other situations. You need only have a one-parameter family $\{F_c\}$ of functions, and a property by means of which they can be classified. A $c$-plane chart $M$ is then defined: $c$ is in $M$ whenever $F_c$ has the property in question. Once a picture of $M$ has been produced, it can be used as a way of locating examples of interest in the family $\{F_c\}$.

Moreover, you have seen how points that are not in the Mandelbrot set can be assigned colors in a meaningful way, thus adding further information to this chart. Similar opportunities may exist in other $c$-plane charts.

Here is an interesting application of this idea: You have seen (item 6 on page 51) that there are cubic polynomials for which the Newton-Raphson method has an attracting cycle. The rarity of such examples leads one to ask, How are they found? It seems to be the proverbial search for a needle in a haystack. The answer is to use a chart!

For each $c$, let $E_c(z) = 4z^3 - (c^2 + 3)z + c^2 - 1$. An important feature common to these cubic polynomials is that $E''_c(0) = 0$, making zero a critical point for the Newton-Raphson processes $N_c$ (see items 2 and 6 on page 266). These polynomials also have known roots, $1$ and $\frac{1}{2}(-1 \pm c)$, to which colors can be assigned. The chart $M$ may now be defined: A parameter $c$ belongs to $M$ if zero is not attracted to any of the roots of $E_c$. Those $c$-values that are not in $M$ can be colored according to the root to which zero is attracted.

This chart $M$ contains exactly the information that is needed; one hopes only that it will be visible. In other words, the (uncolored) components of $M$ may be so small that they are hard to find! It is fortunate that this is not the case. As Figure 77 shows, the critical point zero is attracted to a root for nearly every value of $c$; these three cases are represented by white and two shades of gray. The exceptional cases are represented by black components, one of which has been magnified.

This chart does an effective job of surveying the performance of the Newton-Raphson method for this particular family of cubic equations. One wonders about other cubic equations, however. By concentrating on the functions $E_c$, have any interesting examples been left out?

1. Consider the equation $z^3 - 3z^2 + 2z + 10i = 0$, and let $N$ be the associated root-finding function. The linear function $U(z) = 2iz + 1$ establishes an isomorphism with one of the above systems; namely, $U(N_c(z)) = N(U(z))$. Find $c$.

In general, any cubic equation that has at least two distinct roots is dynamically equivalent to one of the examples $E_c$, by means of a linear isomorphism.
Dynamic Fractals

Figure 77 is centered at the origin of the $c$-plane, and its width is 7. Each $c$-value represents a polynomial, namely $E_c$ as defined on page 54. Recall that the roots of $E_c$ are

$$1, \; \frac{1}{2}(-1 + c), \; \text{and} \; \frac{1}{2}(-1 - c).$$

Colors are assigned according to the behavior of the critical orbit \{0, $N_c(0)$, $N_c^2(0)$, \ldots\} when the Newton-Raphson method is applied — dark gray means that the orbit approaches 1, white means that the orbit approaches $\frac{1}{2}(-1 + c)$, light gray means that the orbit approaches $\frac{1}{2}(-1 - c)$, and black means that the orbit approaches none of the roots of $E_c$. Notice that there are small patches of black in the figure.

Figure 78 is a tenfold magnification of a region just above the center of Figure 77. It includes two of the most prominent blobs on the imaginary axis. Near the top of each blob is a barely visible but familiar-looking patch of black (not dark gray).

Figure 79 is a tenfold enlargement of the largest black patch in Figure 78. Although it may seem surprising to find the Mandelbrot set in the middle of a figure that describes the iteration of a rational function of degree 3, the analysis of item 1 on page 46 has prepared you for this possibility. The unusual polynomial $E_c$ that was examined in Figures 75 and 76 was chosen (rather, $c = 0.007 + 1.019i$ was chosen) from the period-tripling component (index $\theta = 240$) attached to the right side of the cardioid you see. This explains why the rabbit motif crept into the Julia set for this example.

What evidence in Figure 79 tells you that this is not a metrically exact copy of the Mandelbrot set? Given that the current example (number 6 on page 51) was chosen from one of the period-tripling components, and that the critical orbit is attracted to a 9-cycle, what can you say about the examples $E_c$ chosen from the interior of the cardioid, or from the other period-tripling disk?
Dynamic Fractals

Self-similarity (which is a conspicuous feature of Julia sets) is actually lacking from the Mandelbrot set, despite the profusion of miniature Mandelbrot sets that you have seen. A localized version of self-similarity can be found, however, and the place to look for it is at a Misiurewicz point. The Mandelbrot zoom sequence shown in Figure 80 is centered at the Misiurewicz point $c = -0.168987114927887 + 1.0423702532716137i$, which has the same relation to the small period-4 satellite as $-2$ does to the full Mandelbrot set.

The widths of the four frames are 0.2, 0.02, 0.002, and 0.0002, reading clockwise from the upper left frame. It is not hard to believe that the Mandelbrot set slowly spirals counter-clockwise about $c$. It is also believable that a suitable magnification factor would produce virtually no change in the figure. That factor is approximately $3.7375 - 0.3442i = 3.7533 \text{cis}(-5.2611)$. In other words, if the final image were magnified by 3.7533 and turned clockwise through 5.2611 degrees, there would be no evident change.

The source for this data and the self-similarity is a remarkable theorem that says that the magnified Mandelbrot set is virtually indistinguishable from the corresponding Julia set, suitably magnified at $c$ (which belongs to the Julia set because $c$ is a Misiurewicz point). The multiplier is actually the derivative of the eventual cycle (in this case, a 4-cycle). The Julia set of $Q_c$ is shown in the upper left frame of Figure 81. The other three frames are centered at $c$, which is located near the top of the Julia set. In clockwise order, each frame is a 100-fold magnification of the preceding.

Escape-time contours are special curves that surround Julia sets, such as appear in the familiar Figures 19, 20, 21, and 82. They are typically defined and computed with respect to an arbitrarily chosen large circle, and therefore seem to lack any absolute significance. In fact, these discrete contours are just approximations to a continuous system of curves that describes the exterior of a filled-in Julia set in absolute terms.

Each time that $Q_0$ is applied to a $z$-value outside the unit circle, the radial coordinate is squared, which means that the logarithm of the radius is doubled. In other words,

$$\frac{1}{2^n} \log |Q^n_0(z)|$$

depends only on $z$, being equal to just $\log |z|$. There is no reason to expect such simplicity for other functions $Q_c$, but it is reasonable to define the real-valued function

$$G_c(z) = \lim_{n \to \infty} \frac{1}{2^n} \log |Q^n_c(z)|$$

for $z$ outside the filled-in Julia set, and to extend the definition so that $G_c(z) = 0$ for all other $z$ (for which $|Q^n_c(z)|$ does not approach infinity). When $|z|$ is large, the effect of adding $c$ during the computation of $Q_c(z)$ is negligible, so you should expect that the sequence that defines $G_c(z)$ converges rapidly. The contours that you see in Figures 19 and 82 are essentially level curves of $G_c$, obtained by solving $G_c(z) = r$ for various positive values of $r$. Thus $G_c$ gives absolute numerical information about the position of $z$ relative to the Julia set of $Q_c$. It is a good exercise to verify that $G_c(Q_c(z)) = 2G_c(z)$. This result allows you to define the dynamic distance from $z$ to the Julia set by means of

$$H_c(z) = \log_2(G_c(z)),$$

which makes sense for all $z$ exterior to the filled-in Julia set.
Dynamic Fractals

Fractal dust occurs when a Julia set becomes disconnected. When \( c \) is real, with \( c < -2 \), you have seen that the critical orbit escapes to infinity, through the bottom of the box shown at right. In fact, almost all orbits escape to infinity. The Julia set consists of all the orbits that do not escape. As you have seen, this includes all points that are ancestral to \( n \)-cycles. Take \( c = -2.31 \), for example. The fixed points for \( Q(x) = x^2 - 2.31 \) are \( p = 2.1 \) and \( r = -1.1 \), both repelling. Their ancestors provide infinitely many points of the Julia set. The critical orbit escapes the box whose corners are \((p, p), (-p, p), (-p, -p), \) and \((p, -p), \) because \(-2.31 < -2.1 \). The ancestors of \( p \) play a prominent role in the following construction of the Julia set:

(i) Remove from consideration all \( x \)-values strictly between \(-\sqrt{0.21} \approx -0.45826 \) and \(\sqrt{0.21} \approx 0.45826 \). Along with the critical orbit, the orbits of these points leave the box on the first iteration. Notice that the endpoints just introduced are ancestors of \(-p\).

(ii) Let \( h = \sqrt{2.31 - 0.21} \approx 1.36079 \) and \( k = \sqrt{2.31 + 0.21} \approx 1.66381 \). Remove \( x \)-values that are strictly between \(-k\) and \(-h\) or strictly between \(h\) and \(k\). The orbits of these points leave the box on the second iteration. The four endpoints just introduced are ancestral to \( p \). Notice that the ancestors of \( r \) are interior to the intervals that remain.

Repeating this process forever will reduce the interval \(-p \leq x \leq p\) to a dust of points that are never removed. The dust is the Julia set; it includes all the periodic points, their ancestors, and so forth. Notice that the ancestors of \( p \) are the most conspicuous points during this process; indeed, they serve to define the process.

Point sets constructed by this interval-removal method are known as Cantor sets.

1. Calculate approximately the details of the third stage of the interval-removal process.

2. After the \( n^{th} \) stage of the interval-removal process is complete, how many pieces of the interval \(-p \leq x \leq p\) will be left?

3. What is the sum of the lengths of all the intervals removed during the process?

The Cantor process must be modified to deal with the disconnected Julia sets that occur when \( c \) is a real value greater than \( \frac{1}{4} \), for such a Julia set does not consist exclusively of real points. The process can in fact be modified to make it apply to any disconnected quadratic Julia set.
Dynamic Fractals

1. The figure at right is the Julia set of some function $Q_c$. Given that the real members of this set are $\pm \frac{1}{4}$, find the value of $c$, as well as the two members that are pure imaginary.

2. Is it possible for a real quadratic polynomial to have an attracting 3-cycle? Give an example, or else explain why there is no such example.

3. Find a complex number that is part of a 6-cycle for $Q_0$.

4. Find the quadratic function that has a superattracting fixed point at $z = 2$ and another fixed point at $z = 0$. Make an accurate sketch of the Julia set for your polynomial.

5. If a point in the Julia set of $Q_c$ has $\frac{19}{32}$ for its label, then what labels designate its immediate ancestors? How are these two ancestors related geometrically?

6. Give a concise description of the numbers that are used to label eventually fixed members of a quadratic Julia set.

7. Suppose that $m$ is a complex number for which $|m| < 1$, and let $L(z) = mz + b$. Define a sequence recursively by $z_0 = 0$ and $z_{n+1} = L(z_n)$. In terms of $m$ and $b$, find (a) $z_3$; (b) the limit of this sequence.

8. Find a value for $B$ that will make $z = 3$ part of a 2-cycle when the Newton-Raphson method is applied to the equation $Bz - z^3 = 0$.

9. How many 9-cycle components are tangentially attached to the main cardioid of the Mandelbrot set? How many 9-cycle components are there in all?

10. What is the role of the point $\infty$ when the Newton-Raphson method is applied to solve a polynomial equation?

11. It is seldom possible to draw an accurate picture of a Julia set by hand. The polynomial $Q_{-2}$ provides one of the rare examples; the Julia set consists of the real interval $-2 \leq x \leq 2$. Draw this Julia set, and on it mark the points whose dynamic labels are 0, $\frac{1}{8}$, $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{2}$, $\frac{5}{8}$, $\frac{2}{3}$, and $\frac{3}{4}$. You will need to work out coordinates for some of these.

12. It is a remarkable theorem that the basin of attraction of $\infty$ for the quadratic system $Q_c$ is isomorphic to the basin of attraction of $\infty$ for $Q_0$, whenever $c$ is in the Mandelbrot set. The example $c = -2$ is probably the only nontrivial example for which it is possible to write down an explicit isomorphism. Show that the function $\phi(z) = z + z^{-1}$ is an isomorphism, then use this isomorphism to write down a labeling function for the Julia set. This enables you to check your answers to the preceding question.
Dynamic Fractals

1. The IIM was used to draw the rough outline of the Julia set of $Q_{-1}$ shown below. The letters $A \ldots G$ mark some of the components of the basin of the attractive 2-cycle.
(a) In which components is the 2-cycle found?
(b) Tell the destination for each of these letters.
(c) Identify the two field lines that land at the pinch point between components $C$ and $F$.

2. Suppose that $F(p) = p$ and that $F'(p) = \sqrt{3} - i$. Make a dot on your paper to represent $p$, and another dot about $\frac{1}{8}$ inch to the right to represent a nearby point $q$. Now make dots to represent the points $F^n(q)$ for $n = 1, 2, 3, 4$.

3. Figure 83 shows the Julia set for the Newton-Raphson cube-root finder. For each of the ten lettered regions, identify the region to which it is sent by one application of the process.

4. Restricted to the invariant region lettered $h$ in Figure 83, the cube-root process is isomorphic to the squaring process inside the unit circle. Therefore the points on the boundary of $h$ can be given real labels between 0 and 1. What label is assigned to the origin? the point shared by regions $a$ and $b$? the point shared by regions $e$ and $h$?

5. Find a cubic polynomial that has the 3-cycle $0 \rightarrow 1 \rightarrow 2 \rightarrow 0$. Is this cycle attracting for your polynomial?
Dynamic Fractals

Isomorphism and coordinates.

The point at infinity is superattracting for any quadratic polynomial. Its basin of attraction (the complement of the filled-in Julia set) is invariant, and of special interest because any two such basins are isomorphic whenever the associated Julia sets are connected (i.e., whenever the $c$-values are chosen from the Mandelbrot set). In particular, all such examples are isomorphic to $Q_0$ acting on the exterior of the unit disk. Moreover, the isomorphism is unique, determined by the example itself. A comparison of the figures strongly suggests that every point exterior to the filled-in Julia set in Figure 19 has a uniquely determined dynamic counterpart that is exterior to the unit circle in Figure 82.

Although it is almost never possible to write down the correspondence in finite terms, there is an intuitive approach to the problem that does yield useful approximations. Let $\Phi$ denote the desired isomorphism between $Q_c$ and $Q_0$, so that $\Phi(Q_c(z)) = Q_0(\Phi(z)) = [\Phi(z)]^2$ holds for all $z$ that are outside the filled-in Julia set of $Q_c$. In particular, $z$ and $\Phi(z)$ approach infinity at the same rate when acted on by $Q_c$ and $Q_0$, respectively, so it seems reasonable, when $n$ is large, that

$$Q^n_c(z) \approx \Phi(Q^n_c(z)) = Q^n_0(\Phi(z)) = [\Phi(z)]^{2^n}.$$  

One reason for expecting this is that the addition of $c$ during each iteration is eventually insignificant compared to the very large $z$-values; thus you might as well be squaring repeatedly. It is therefore tempting to solve for $\Phi(z)$ by simply taking $n$ successive square roots:

$$\Phi(z) \approx \sqrt[n]{\ldots \sqrt[2]{Q^n_c(z) = [Q^n_c(z)]^{(1/2)^n}}}$$

On the next page, the subscript $c$ is dropped from $Q_c$, to unclutter the notation.
Dynamic Fractals

To be precise, you might try defining \( \Phi(z) \) to be the limit of the preceding, as \( n \) approaches infinity. The trouble with all of this, however, is that root extraction is an ambiguous process (there are always two roots to choose from), hence the right-hand side of the display has \( 2^n \) possible meanings, only one of which is actually correct.

It is helpful to rewrite this expression as a telescoping product, as shown next. Notice that \( Q^n(z)^m \) means the \( m \)th power of the \( n \)-fold application of \( Q \) to \( z \).

\[
\Phi(z) \approx z \cdot \frac{Q(z)^{1/2}}{z} \cdot \frac{Q^2(z)^{1/4}}{Q(z)^{1/2}} \cdot \frac{Q^n(z)^{(1/2)^n}}{Q^{n-1}(z)^{(1/2)^{n-1}}} \\
= z \cdot \left( \frac{Q(z)}{z^2} \right)^{1/2} \left( \frac{Q^2(z)}{Q(z)^2} \right)^{1/4} \cdots \left( \frac{Q^n(z)}{Q^{n-1}(z)^2} \right)^{(1/2)^n} \\
= z \cdot \left( 1 + \frac{c}{z^2} \right)^{1/2} \left( 1 + \frac{c}{Q(z)^2} \right)^{1/4} \cdots \left( 1 + \frac{c}{Q^{n-1}(z)^2} \right)^{(1/2)^n}
\]

Because the approximation improves as \( n \) grows, it is reasonable to define

\[
\Phi(z) = z \prod_{n=1}^{\infty} \left( 1 + \frac{c}{Q^{n-1}(z)^2} \right)^{(1/2)^n}.
\]

As long as \( |c| < |Q^n(z)|^2 \) holds for all nonnegative \( n \), \( \Phi(z) \) is well defined, for none of the indicated square roots has a negative real part. In other words, this defines \( \Phi(z) \) for all \( z \) that are sufficiently large (at least for \( 2 < |z| \)). What about other values of \( z \)? Advanced theory says — when the Julia set of \( Q \) is connected — that the square roots in the above formula can be chosen to define \( \Phi(z) \) for all \( z \) outside the filled-in Julia set.

The key property of the resulting function \( \Phi \) is that \( \Phi(z^2) = \Phi(Q(z)) \). It follows that \( \Phi(z)^{2^n} = \Phi(Q^n(z)) \). Because \( \Phi(Q^n(z)) \) and \( Q^n(z) \) are virtually the same for large \( n \), this shows that \( G(z) = \log |\Phi(z)| \) is the escape-time function defined on page 56. The function \( \Phi \) thus introduces a canonical system of coordinates into the exterior of the filled-in Julia set of \( Q \), consisting of escape-time contours and the orthogonal family of field lines, whose associated indices you have already met in discussions about the Julia set. For rational indices, these field lines point at definite members of the Julia set. This is not always the case with irrational indices, however. Such a ray can behave so erratically when approaching the Julia set that it does not really land at any definite place. This sort of thing happens for the Julia set shown Figure 35, for example (but not for the Julia sets shown in Figures 32, 33, and 34).
Dynamic Fractals

The Distance-Estimator Method is based on the following simple yet remarkable theorem of complex analysis, due to P. Koebe:

Suppose that $F$ is a one-to-one and differentiable complex function defined on the unit disk $D : |z| < 1$. Let $r = |F'(0)|$ and $b = F(0)$. The distance from $b$ to the boundary of $F(D)$ is at least $\frac{1}{4} r$.

1. Consider the example $F(z) = z - \frac{1}{2} z^2$. Show that $F$ maps the unit disk interior $|z| < 1$ in a one-to-one manner. Verify the conclusion of the theorem.

2. Given complex constants $a, b, c, d$, with $ad \neq bc$, the function

$$L(z) = \frac{az + b}{cz + d}$$

is called linear fractional. Two examples were considered in item 5 on page 36. Establish the following properties of $L$ (viewed as a transformation of the complex sphere):

(a) $L$ is one-to-one.
(b) $L(\infty) = a/c, L(0) = b/d, L(-d/c) = \infty$, and $L(-b/a) = 0$.
(c) Why are the cases in which $ad = bc$ excluded?
(d) If $a = 1 = d$ and $c = \overline{b}$ (the conjugate of $b$), then $L$ maps the unit circle to itself. If $|b| < 1$ as well, then the unit disk is mapped to itself in a one-to-one manner.

Given a point $p$ whose $Q_c$-orbit approaches $\infty$, it is desirable to calculate the distance from $p$ to the nearest point in the Julia set of $Q_c$. This can be done with the aid of some auxiliary functions, one of which has already been introduced, on page 60 — the isomorphism $\Phi$ from the attracting basin of infinity to the exterior of the unit disk. (As on page 60, the subscripts $c$ are suppressed below, to simplify the notation.) It follows that $1/\Phi$ maps the infinite basin onto the interior of the unit disk, with $p$ sent to some nonzero point $q = 1/\Phi(p)$, and $\infty$ sent to 0. The function inverse to $1/\Phi$ maps $q$ to $p$ and 0 to $\infty$, in a one-to-one fashion. To apply the theorem of Koebe, however, you need a mapping of the open unit disk that sends 0 to $p$ and that sends no interior point to $\infty$.

3. Consider the linear fractional $L(z) = \frac{z + q}{1 + \overline{q}z}$. Show that $L$ maps the unit disk onto itself, with 0 sent to $q$ and $-q$ sent to 0, then show that $M(z) = L(qz)$ maps the unit disk into itself in a one-to-one fashion, with 0 sent to $q$ and $-1$ sent to 0.

Now use the linear fractional $M$ just defined to build the desired function. Let

$$T(z) = \Psi \left( \frac{1}{M(z)} \right) = \Psi \left( \frac{1 + \overline{q}z}{qz + q} \right),$$

where $\Psi = \Phi^{-1}$. $T$ is a one-to-one function that maps the unit disk into the attracting basin of $\infty$, with 0 sent to $p$ and $-1$ sent to $\infty$. Calculate

$$T'(0) = \Psi' \left( \frac{1}{q} \right) \frac{q\overline{q} - 1}{q} = \left( \frac{1}{\Phi(p)} - \Phi(p) \right) \frac{1}{\Phi'(p)}.$$
Dynamic Fractals

Thus
\[
\frac{1}{4} |T'(0)| = \left| \frac{1 - \Phi(p)\Phi(p)}{4\Phi(p)\Phi'(p)} \right| = \frac{|\Phi(p)|^2 - 1}{4|\Phi(p)||\Phi'(p)|}.
\]

Next use the relation \(G(z) = \log|\Phi(z)|\) from page 61 to obtain
\[
\frac{1}{4} |T'(0)| = \frac{\sinh(G(p))}{2|\Phi'(p)|}.
\]

The theorem of Koebe asserts that the distance from \(p\) to the Julia set is \textit{at least} this amount. Because overestimating the actual distance would be a serious mistake (why?), it is convenient to proceed by removing the sinh function and writing
\[
\frac{\log|\Phi(p)|}{2|\Phi'(p)|} < \frac{1}{4} |T'(0)|.
\]

In other words, \(\frac{\log|\Phi(p)|}{2|\Phi'(p)|}\) is an underestimate of the distance from \(p\) to the nearest point on the Julia set. To calculate the underestimate, consider the orbit \(\{z_n\}\) of \(p\). In other words, let \(z_0 = p\) and \(z_n = Q(z_{n-1})\) for all positive values of \(n\), and assume that \(z_n \to \infty\).

Because (see page 60)
\[
|\Phi(p)| = \lim_{n \to \infty} |Q^n(p)|^{2^{-n}},
\]

it follows that
\[
\log |\Phi(p)| \approx \frac{1}{2^n} \log |Q^n(p)| = \frac{\log |z_n|}{2^n}
\]

and
\[
|\Phi'(p)| \approx \frac{1}{2^n} |Q^n(p)|^{2^{-n} - 1} |Q'(p) \cdot Q'(Q(p)) \cdot Q'(Q^2(p)) \cdots Q'(Q^{n-1}(p))| = \frac{1}{2^n} |z_n|^{2^{-n} - 1} |2z_0 \cdot 2z_1 \cdot 2z_2 \cdots 2z_{n-1}|.
\]

Therefore
\[
\frac{|z_n| \log |z_n|}{2^{n+1} |z_0 z_1 z_2 \cdots z_{n-1}| |z_n|^{2^{-n}}} \approx \frac{\log |\Phi(p)|}{2|\Phi'(p)|}
\]

underestimates the distance from \(p\) to the nearest point of the Julia set, provided that \(n\) is large enough. Notice that the distance estimator for \(p\) can be calculated while the orbit itself is being calculated. Notice also that there is essentially no change in the estimator value once \(n\) is large enough to make \(z_{n+1} \approx z_n^2\).
**Dynamic Fractals**

**Indifferent 3-Cycles.**

Every function $Q_c$ has two 3-cycles, which can be found by solving the equation

$$z = Q_c^3(z) = ((z^2 + c)^2 + c)^2 + c$$

or

$$0 = z^8 + 4cz^6 + (6c^2 + 2c)z^4 + (4c^3 + 4c^2)z^2 - z^4 + c^4 + 2c^3 + c^2 + c.$$  

(1)

Two of the roots are fixed points of $Q_c$, and thus satisfy $z^2 + c - z = 0$. The six 3-cycle points are defined by the equation that results from dividing equation (1) by $z^2 - z + c$, namely

$$0 = z^6 + z^5 + (3c + 1)z^4 + (2c + 1)z^3 + (3c^2 + 3c + 1)z^2 + (c + 1)^2z + c^3 + 2c^2 + c + 1.$$  

(2)

The boundary of a 3-cycle component consists of those $c$ for which the 3-cycle is *indifferent*, which means that

$$1 = \left| \frac{d}{dz} Q_c^3(z) \right| = |2z \cdot 2Q_c(z) \cdot 2Q_c^2(z)| = |2z \cdot 2(z^2 + c) \cdot 2((z^2 + c)^2 + c)|.$$  

Thus, for $c$ to be on the boundary of a 3-cycle component, it is necessary that

$$0 = z^7 + 3cz^5 + (3c^2 + c)z^3 + (c^3 + c^2)z - v,$$  

(3)

where $8v$ is a unit complex number, and $z$ is one of the six 3-cycle points of $Q_c$. Because $c$ and $z$ satisfy equation (2), they must also satisfy the difference between equation (3) and $z$ times equation (2), which is

$$0 = z^6 + z^5 + (2c + 1)z^4 + (2c + 1)z^3 + (c + 1)^2z^2 + (c^2 + c + 1)z + v.$$  

By forming combinations in this fashion, with the intent of creating equations of lower degree in $z$, you will eventually discover that

$$0 = (c^3 + 2c^2 + c - cv + 1 - 2v + v^2) (z^2 + z + c),$$

which is the difference between $z^4 - z^2 + cz^2 - z + 1$ times equation (2) and $z^3 + z^2 + cz - 1$ times equation (3). If $z^2 + z + c$ were 0, then $(-z)^2 + z + c$ would also be 0, and $-z$ would be one of the fixed points of $Q_c$. This is impossible, however, because $Q_c(z)$ would equal $-z$, and $z$ is one of the 3-cycle points for $Q_c$. Thus $z^2 + z + c$ is not 0. It follows that

$$0 = c^3 + 2c^2 + c - cv + 1 - 2v + v^2 = c^3 + 2c^2 + (1 - \frac{u}{8})c + \left(1 - \frac{u}{8}\right)^2,$$  

(4)

where $u$ is a unit complex number, defines the boundaries of the three Mandelbrot 3-cycle components. There are three values of $c$ for each value of $u$. 

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May 2006  
64  
Phillips Exeter Academy
Dynamic Fractals

In particular, consider $u = 1$. Equation (4) becomes $0 = 64c^3 + 128c^2 + 56c + 49$, which can be factored as $0 = (4c + 7)(16c^2 + 4c + 7)$. One root is $c = -\frac{7}{4}$, which is the cusp of a satellite cardioid, and the other two are $c = \frac{1}{8} (1 - \pm i\sqrt{27}) \approx -0.125 \pm 0.64951905i$, which are the points where two 3-cycle components are attached to the main cardioid.

Next consider $u = -1$. Equation (4) becomes $0 = 64c^3 + 128c^2 + 72c + 81$, whose roots can be found by using the cubic formula. The real one is

$$c = -\frac{2}{3} - \frac{1}{24} \sqrt[3]{7660 + 540\sqrt{201}} - \frac{1}{24} \sqrt[3]{7660 - 540\sqrt{201}} \approx -1.76852915,$$

which is the leftmost point on the satellite cardioid. The nonreal roots are

$$c = -\frac{2}{3} + \frac{1}{48} \sqrt[3]{7660 + 540\sqrt{201}} + \frac{1}{48} \sqrt[3]{7660 - 540\sqrt{201}}
\pm \frac{1}{48} \left( \sqrt[3]{7660 + 540\sqrt{201}} - \sqrt[3]{7660 - 540\sqrt{201}} \right) i\sqrt{3}
\approx -0.11573542 \pm 0.83799903i.$$

All three roots are bifurcation points where 3-cycle components are tangent to 6-cycle components.

When $u = i$, the three indifferent points are

$$-1.7504806116 + 0.0097757841i$$
$$-0.0303264157 + 0.7411028161i$$
$$-0.2191929727 - 0.7598786002i$$

and when $u = -i$, the indifferent points are the conjugates of the preceding.

**Not every Mandelbrot component is bounded by a circle or a cardioid**

For example, consider the 3-cycle component whose Mandelbrot center is approximately $-0.12256117 + 0.74486177i$ (from page 244), and whose boundary contains the indifferent points

$$p_1 = -0.125 \quad + 0.64951905i$$
$$p_2 = -0.03032642 + 0.74110282i$$
$$p_3 = -0.11573542 + 0.83799903i$$
$$p_4 = -0.21919297 + 0.75987860i$$

The center of the circle that goes through $p_1, p_2,$ and $p_3$ is $-0.12473794 + 0.74397386i$. The center of the circle that goes through $p_2, p_3,$ and $p_4$ is $-0.12600747 + 0.74403626i$, however, showing that $p_1, p_2, p_3,$ and $p_4$ are not concyclic.
The Gingerbread Man.

The title refers to configurations that arise when the nonlinear two-dimensional recursion

\[
\begin{align*}
x_{n+1} &= 1 + |x_n| - y_n \\
y_{n+1} &= x_n
\end{align*}
\]

is applied to initial points \( P_0 = (x_0, y_0) \). The figure at right shows the first 210 terms of the sequence that begins at \( P_0 = (1.6, 0.4) \). Because \( P_{210} = P_0 \), the sequence is periodic, and there is actually no more to see.

Indeed, any sequence that begins at a point that has rational coordinates is periodic, though this is far from obvious! What one can prove (with some difficulty) is that every sequence generated by this recursion is bounded, meaning that it is contained within a suitably large rectangle. Periodicity then follows from this fact. How?

The period of the sequence that begins at \((1.601, 0.401)\) is 35130. This is strikingly different from the sequence that begins at the nearby point \((1.6, 0.4)\). It is also interesting to compare these two sequences term-by-term. One finds, for example, that \( P_{137} = (-0.415, 0.484) \) for one of the sequences and \( P_{137} = (-2.8, 3.8) \) for the other. (Can you tell which is which?) This illustrates how solutions can be extremely sensitive to initial conditions. A slight change in the initial data of a problem can lead to dramatically different results.

Though this sensitivity is often encountered in nonlinear differential equations, it is not the rule, nor is it easy to detect even when it is present. For example, the sequences that begin at \( P_0 = (0.8, 1.6) \) and at \( P_0 = (0.799, 1.601) \) behave exactly the same — both have period 6. This period is in fact obtained for any sequence whose initial point is chosen within the hexagon whose corners are \((0, 0), (1, 0), (2, 1), (2, 2), (1, 2), \) and \((0, 1)\), with the exception of the central point at \((1, 1)\), whose period is 1.

Choose any non-central point inside the hexagon whose corners are \((2, 0), (2, -1), (3, -2), (4, -2), (4, -1), \) and \((3, 0)\). You should find that the period of the resulting sequence is 30. What is the period of the sequence that starts at the center of the hexagon?

Sensitivity to initial conditions can cause gross errors in computer-generated pictures. Whenever a machine calculation requires rounding (as most do), all subsequent calculations are suspect. For example, a calculation that has been initialized at \((1.6, 0.4)\) is off to a bad start, because a computer’s internal binary scale makes it impossible to store even these coordinates accurately! (Try writing \(1/10\) in terms of powers of \(1/2\).) As the 3200 dots at right show, the calculated points \( P_n \) become increasingly remote from the correct points \( P_n \). Thus, not only does the computer suggest that the sequence is aperiodic, it actually makes it look as if the sequence will fill sections of the plane.
Reference
**Dynamic Fractals Reference**

**2-cycle:** A cycle whose period is 2. [12,17,25,28,218,222,234,247]

**ancestor:** Given a dynamic system $z_n = f(z_{n-1})$, a point $p$ is called an ancestor of a point $q$ if $q$ appears in the orbit of $p$. In other words, $q = f^n(p)$ for some positive integer $n$. If $q = f(p)$, then $p$ is an immediate ancestor of $q$, also known as an inverse image of $q$. The words “parent” and “grandparent” are also descriptive and useful. [10,21]

**ancestral tree:** The set of all ancestors of a given point. This genealogical term also suggests a useful organizational hierarchy that the term “backwards orbit” (which appears in some books) does not. [23]

**aperiodic:** Applied to a sequence of values that is neither periodic nor eventually periodic. For example, the decimal expansion of an irrational number is aperiodic. [19]

**argument:** In a mathematics book, this word simply refers to any input for a function. In a book about complex variables, $e^{i\theta}$ is a most significant function, and thus the polar angle $\theta$ is often called “the argument.” [218]

**attracting fixed point:** Given a dynamic system $z_n = f(z_{n-1})$, a fixed point $p$ is called attracting if $|f'(p)| < 1$. This means that all orbits that originate suitably close to $p$ will converge to $p$. [9]

**attracting cycle:** Given a dynamic system $z_n = f(z_{n-1})$, a $k$-cycle $\{p_1, p_2, \ldots, p_k\}$ is called attracting if $|f'(p_1)||f'(p_2)| \cdots |f'(p_k)| < 1$. This is equivalent to saying that $p_1$ is an attracting fixed point for $f^k$, and it means that all orbits that originate suitably close to $p_1$ will approach $\{p_1, p_2, \ldots, p_k\}$ as a limiting cycle. [9]

**Babylonian method:** Also known as the “divide-and-average” method, this is how Babylonians who needed square roots found them. [1]

**basin of attraction:** Given a dynamic system $z_n = f(z_{n-1})$, and an attracting fixed point $p$ for $f$, the basin of attraction of $p$ is the union of all orbits that approach $p$. [221]

**bifurcation:** A splitting in two. In this book, the terminology refers to the birth of an attracting $2k$-cycle from a formerly attracting $k$-cycle. [42,43,263]

**Cantor Set:** Georg Cantor (1845—1918) constructed his “middle-thirds” set by taking a line segment, removing the middle third of it, then recursively removing the middle thirds of all intervals encountered thereafter. The terminology is now extended to similar constructions, whether or not they begin with an actual interval, or remove according to a fixed percentage. [68]

**cardioids:** The boundary of the Mandelbrot set includes infinitely many quasi-circles and quasi-cardioids. One of the quasi-cardioids is a true cardioid. [32,38,243,247]

**catalogue:** A subset of the parameter plane that classifies a one-parameter family of dynamic systems in the $z$-plane. The best-known example is the Mandelbrot set, which classifies quadratic Julia sets. [52]
Cayley’s problem: To understand the behavior of the Newton-Raphson method when it is applied to complex polynomials. Proposed by Arthur Cayley (1821—1895) in 1879. [221]

center of Mandelbrot component: A $c$-value for which $Q_c(z) = z^2 + c$ has a super-attractor. These $c$-values are found at the “centers” of Mandelbrot components (disk or cardioid). [249]

centers of 3-cycle components are tabled on page 244.

centers of 4-cycle components are tabled on page 35.

centers of 5-cycle components are tabled on page 35.

centers of 6-cycle components are tabled on page 36.

Chain Rule: If $f$ and $g$ are differentiable functions, and if $h(z) = f(g(z))$, then $h$ is also differentiable, and $h'(z) = f'(g(z))g'(z)$. [38,253,258]

chaos: There is no generally accepted technical meaning of this word, which usually suggests total disorder. Applied to dynamic systems, it refers to the erratic behavior of points within the invariant Julia set — specifically to sensitivity on initial conditions, which means that orbits that start very close together are destined to be very far apart.

circles: The boundary of the Mandelbrot set includes infinitely many quasi-circles and quasi-cardioids. One of the quasi-circles is a true circle. [76,247]

closed set: A set that includes all its limit points. [38]

closure: The closure of a set $S$ consists of $S$ together with all the limit points of $S$. It can be proved that the closure of $S$ is actually a closed set.

completely invariant: Given a dynamic system, a set is called completely invariant if it contains the orbit and the ancestral tree of each of its points. [23]

complex numbers: Numbers of the form $a + bi$, where $a$ and $b$ are ordinary real numbers. The real-number system is a subsystem of the complex-number system. [2]

complex sphere: The result of adding a “point at infinity” to the complex plane. See stereographic projection. [18]

conformal: angle-preserving [18]

critical orbit: Given a dynamic system $z_n = f(z_{n-1})$ defined by a differentiable function $f$, and given a critical point $p$ of $f$ (which means that $f'(p) = 0$), the sequence $p, f(p), f(f(p)), \ldots$ is called a critical orbit. For quadratic examples $Q_c(z) = z^2 + c$, the only critical value is 0, so the only critical orbit is 0, $c, c^2 + c, (c^2 + c)^2 + c, \ldots$ [32,33,39,68,252,262]

Critical Orbit Theorem: If $p$ is an attracting fixed point for a rational complex function $F$, then $p$ must attract the orbit of at least one critical point of $F$. [32]
Dynamic Fractals Reference

cubic convergence: If a sequence \( \{z_n\} \) approaches a limiting value \( z_\infty \), and if there is a positive constant \( k \) for which \( |z_n - z_\infty| \leq k|z_{n-1} - z_\infty|^3 \) holds for all large \( n \)-values, the convergence is called “cubic.” [207] For example, Hutton’s method of root-finding produces sequences that approach their limits cubically.

cubic formula: Analogous to the quadratic formula, this expresses the roots of an arbitrary cubic equation in terms of the coefficients of the equation, using square roots and cube roots. It is rather complicated. [13]

cycle: Given a dynamic system, a finite (periodic) orbit is called a cycle. It is called a \( k \)-cycle if its period is \( k \). A 1-cycle is just a fixed point.

DeMoivre’s Theorem: Describes how to raise complex numbers to integer powers: 
\[
(r e^{i\theta})^n = r^n e^{i n \theta}
\] [208]

dendrite: Describes the branched, treelike appearance of connected Julia sets whose \( c \)-values are found on the filaments of the Mandelbrot set. [41,43]

dense: A set \( S \) is dense in another set \( T \) when \( S \) is a subset of \( T \), and the closure of \( S \) is \( T \). In other words, every point of \( T \) has points of \( S \) that are arbitrarily close to it. An important theorem of Julia states that the ancestral tree of any point in a Julia set is a dense subset of the Julia set. [19,253]

Distance-Estimator Method (DEM): A method of drawing the Mandelbrot set and quadratic Julia sets. [45,54,73,250]

dragon: Describes a quadratic Julia set whose \( c \)-value is found within the cusp of the main cardioid of the Mandelbrot set. [48,160]

dyadic rational: A rational number of the form \( \frac{m}{2^n} \), where \( m \) and \( n \) are integers. [24,231]

dynamic distance: The value of a real-valued function that quantifies how far it is, in iterative steps, from a filled-in quadratic Julia set to an exterior point. The level curves of this function are escape-time contours. [67] They are perpendicular to the field lines.

dynamic plane: Also called the z-plane, this is where sequences, periodic points, and Julia sets are found. Not to be confused with the parameter plane, also known as the c-plane, which is where the Mandelbrot set is found. [62]

dynamic system: Consists of a function that can be used to generate sequences (orbits) recursively. [1]

dynamically related: Two points are dynamically related if they are both ancestors of the same point — in other words, if their orbits intersect. [237]

dynamically equivalent systems: Given two functions \( f \) and \( g \), the dynamic systems defined by \( f \) and \( g \) are called equivalent if there is a third function \( \phi \), assumed one-to-one, that matches corresponding elements. This means that \( f(\phi(z)) = \phi(g(z)) \) for all relevant values of \( z \). [28,50,71,226,256,257]
Dynamic Fractals Reference

**escape threshold**: Same as *infinite threshold*. [28]

**escape-time contours**: Points that are at the same iterative distance from a quadratic filled-in Julia set form an escape-time contour. [67]

**Euler identity**: $e^{i\theta} = \cos \theta + i \sin \theta$. [2]

**eventually fixed point**: A point that is not fixed, but whose orbit contains a fixed point. [12,17,46,223,232]

**eventually periodic point**: A point that is not periodic, but whose orbit contains a periodic point. [46,225,232]

**fat rabbit**: Describes either of the two quadratic Julia sets whose $c$-values are the points of tangency where 3-cycle components meet the main cardioid of the Mandelbrot set. [147]

**field lines**: A system of lines that help to coordinatize the exterior of a filled-in, connected, quadratic Julia set. [24,28,242] These lines are perpendicular to the system of escape-time contours. The terminology is borrowed from electrostatics, where the lines show the direction of the force acting on a particle in an electric field. The escape-time contours are the lines of equipotential.

**filaments**: Also known as “hairs”, these are the parts of the Mandelbrot set that are not disk-like or cardioid-like. The filaments are what tie the Mandelbrot set into a connected whole. They are not really the one-dimensional “arcs” they seem to be, however, for each filament actually skewers infinitely many miniature Mandelbrots (probably densely packed). [43,45,53,250]

**filled-in Julia Set**: A Julia set that is connected typically separates (is the boundary of) two or more *basins of attraction*. In the traditional pictures of quadratic Julia sets, only the basin of infinity is colored — the points that are left black constitute a filled-in Julia set. [21,33]

**finite orbit**: An orbit that has only finitely many different values in it. [225]

**fixed point**: Given a dynamic system defined by a function $f$, a fixed point is a point $p$ for which $f(p) = p$. [3]

**fractal**: A word invented by Mandelbrot to describe strange sets whose dimensionality cannot be described by a whole number — they are said to have “fractional dimension”. For example, there is a theory of dimension that assigns the value $\log_3 4 = 1.26\ldots$ to the Koch snowflake curve. [31,221]

**fractal dust**: When $c$ is a complex number that is not in the Mandelbrot set, the Julia set of $Q_c(z) = z^2 + c$ is a totally disconnected set, which is nevertheless infinite because it contains all the periodic points of $Q_c$. It is a Cantor set. [68]

**Fundamental Theorem of Algebra**: Every complex polynomial has roots. [124]
**geometric sequence**: A sequence of numbers \( a, am, am^2, \ldots \) that is generated recursively by applying a fixed multiplier \( m \). \([31,44]\)

**geometric series**: The sum of a geometric sequence. If the magnitude of the multiplier \( m \) is smaller than 1, then the sum of the infinite geometric series \( a + am + am^2 + \cdots \) is \( \frac{a}{1 - m} \). \([24,236]\)

**Hutton’s method**: A root-finding algorithm that converges cubically. \([107]\)

**image**: If \( q = f(p) \), then \( q \) is sometimes called the image of \( p \). If \( S \) is a set of \( p \)-values, then \( f(S) \) is often used (although it is an abuse of notation) to denote the set of all values \( f(p) \) formed from \( p \) in \( S \), and called the image of \( S \).

**immediate ancestor**: See ancestor.

**index**: An overused word in mathematics. In this book, the word refers to a number between 0 and 1 that is used to describe a point in a quadratic Julia set. The index of the right-hand fixed point is either 0 or 1. \([24,28,234,241]\) A table of 6-cycle indices is found on page 155.

**indifferent fixed point**: A fixed point that is neither attracting nor repelling. \([9,23]\)

**indifferent cycle**: A cycle that is neither attracting nor repelling. \([9,23]\)

**infinity**: Accorded quasi-numerical status in the complex-number system. Represented by a single point (usually the north pole) on the complex sphere. \([12,18,225]\)

**infinite threshold**: A sequence that approaches infinity must eventually stay inside every region of the form \( r < |z| \), no matter how large \( r \) is. These regions are accordingly called “neighborhoods of infinity.” In quadratic Julia-set calculations, a fixed “threshold” value of \( r \) can be chosen, because convergence to infinity is assured once this value is passed by \( |z_n| \). \([28,236]\)

**invariant set**: Given a dynamic system, a set is called invariant if it contains the orbit of each of its points. \([9,242]\)

**inverse image**: Also known as an immediate ancestor. \([10]\)

**Inverse-Image Method (IIM)**: A method for drawing quadratic Julia sets, which plots only points in the Julia set. \([33,54]\)

**isomorphic**: A general term used to describe mathematical structures that are in some sense equivalent. In this book, the word is applied to dynamically equivalent systems. \([28,50,71,226,256,258]\)

**iterated-function notation**: It is customary to use exponential notation for functional iteration: \( f^2(x) \) means \( f(f(x)) \) and \( f^5(x) \) means \( f(f^4(x)) \), and so forth. This is consistent with the usual notation \( f^{-1} \) for the function that is inverse to \( f \). \([114]\)
Dynamic Fractals Reference

iteration ceiling: One can never compute all the terms of an infinite sequence — it is necessary to stop and reach conclusions based on a finite amount of data. A preset maximum number of calculated terms is called the iteration ceiling. [33,54,55,253]

Julia promenade: A one-parameter family of Julia sets that is obtained by choosing a smooth path in the Mandelbrot plane. [45]

Julia set: Given a rational function $R(z)$ of a complex variable $z$, the Julia set of the dynamic system $z_n = R(z_{n-1})$ is the closure of the set of all repelling periodic points of $R$. It is the common boundary of all basins of attraction. [38,221] While recovering from wounds received in World War I, Gaston Julia (1893—1978) wrote a seminal paper on the dynamic properties of rational complex functions, in which he solved Cayley’s problem.

Koebe’s Theorem: If $F$ is a one-to-one, differentiable complex function defined on the unit disk $D$, the distance from $F(0)$ to the boundary of $F(D)$ is at least $\frac{1}{4}\frac{|F'(0)|}{4}$. [73]

land: A field line lands on the Julia set if it approaches a definite point. [24,29]

left-hand fixed point: Unless $\frac{1}{4} \leq c$ is real, one of the two solutions of $z^2 + c = z$ precedes the other, when ordered by real parts. The first root is sometimes an attracting fixed point of the dynamic system defined by $Q_c(z) = z^2 + c$. [30]

limit point: Given a set of points $S$, a point $p$ is a limit point of $S$ if there is a sequence in $S$ that converges to $p$. [31]

linear convergence: If a sequence $\{z_n\}$ approaches a limiting value $z_\infty$, and if there is a positive constant $k$ for which $|z_n - z_\infty| \leq k|z_{n-1} - z_\infty|$ holds for all large $n$-values, the convergence is called “linear”. This statement usually implies that the convergence is not quadratic. For example, the geometric sequence $5 + 4 \left(\frac{2}{3}\right)^n$ approaches 5 linearly (use $k = \frac{2}{3}$). [104]

Mandelbrot set: This consists of all complex numbers $c$ for which the critical orbit does not approach infinity. Equivalently, it consists of all complex numbers $c$ for which the Julia set of $Q_c(z) = z^2 + c$ is connected. [33] Benoit Mandelbrot (1924—), who once rejected the work of Julia, eventually rescued it from oblivion. He also coined the word fractal and discovered the set that bears his name.

Misiurewicz point: A complex number $c$ for which the critical orbit is finite but includes 0 only once. These are points of the Mandelbrot set that are branch points or the ends of filaments. [46]

multiple root: If $f(z)$ is a polynomial function, then $a$ is a multiple root of $f$ if $(z - a)^2$ is a factor of $f(z)$. An equivalent formulation is that $f(a)$ and $f'(a)$ are both zero. [6]

multiplier: In the analysis of a dynamic system, this describes how derivatives are used. [138,244,246,248,251,262]
**Dynamic Fractals Reference**

**Newton-Raphson method**: A method for solving equations by setting up a dynamic system in which the roots are superattractors. It contains the Babylonian square-root method as a special case. Isaac Newton (1642—1727) invented differential calculus, and devised the method as an application. His contemporary Joseph Raphson (1648—1715) published it. [6,57]

**orbit**: Given a function $f$, the orbit of $w$ is the recursively defined sequence $w, f(w), f(f(w)), \ldots$ [15]

**parameter plane**: Given a one-parameter family of dynamic systems $z_n = R_c(z_{n-1})$, this is the set of all $c$-values of interest. Also known as the $c$-plane or the catalog space. [62]

**periodic**: A sequence $\{z_n\}$ is called periodic if $z_k$ equals $z_0$ for some positive integer $k$. The smallest such $k$ is called the period of the sequence. [4,20,225,232]

**period-doubling route to chaos**: The name given to the family of quadratic examples $Q_c$ that result from letting $c$ descend from $c = 0$ to $c = -1.401155 \ldots$ [42] There are in fact infinitely many period-multiplying routes to chaos. [44]

**period-multiplying**: The effect of moving $c$ from one component of the Mandelbrot set into a smaller attached component. An attracting $km$-cycle temporarily merges with a repelling $km$-cycle to form an indifferent $k$-cycle, which then splits apart into an attracting $km$-cycle and a repelling $k$-cycle. For this transition to take place, the Julia set must pinch the original $k$-cycle and all its ancestors. [43]

**period-tripling**: See trifurcation.

**pinch**: Describes what Julia sets do as they undergo a period-multiplying transformation. [40,237,241,252]

**quadratic convergence**: If a sequence $\{z_n\}$ approaches a limiting value $z_\infty$, and if there is a positive constant $k$ for which $|z_n - z_\infty| \leq k|z_{n-1} - z_\infty|^2$ holds for all large $n$-values, the convergence is called “quadratic”. For example, the Newton-Raphson root-finding method produces sequences that approach their limits quadratically. [101]

**quadratic formula**: The roots of $az^2 + bz + c = 0$ are $z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

**rabbit**: Describes a quadratic Julia set whose $c$-value is found within either of the two 3-cycle components of the Mandelbrot set. [40,58,59,61,244,246]

**repelling fixed point**: Given a dynamic system $z_n = f(z_{n-1})$, a fixed point $p$ is called repelling if $1 < |f'(p)|$. Any orbit that originates near $p$ (but not at $p$) will be driven away from $p$. [9]

**repelling cycle**: Given a dynamic system $z_n = f(z_{n-1})$, a $k$-cycle $\{p_1, p_2, \ldots, p_k\}$ is called repelling if $1 < |f'(p_1)||f'(p_2)|\cdots|f'(p_k)|$. This is equivalent to saying that $p_1$ is a repelling fixed point for $f^k$. Any orbit that originates near $p_1$ (but not at $p_1$) will be driven away from $\{p_1, p_2, \ldots, p_k\}$. [9]
Dynamic Fractals Reference

**repelling periodic point**: Given a dynamic system, a point is called a repelling periodic point if it initiates a *repelling cycle*. The repelling periodic points are dense in the Julia set of a complex rational function. [38]

**right-hand fixed point**: Unless $\frac{1}{4} \leq c$ is real, one of the two solutions of $z^2 + c = z$ precedes the other, when ordered by real parts. The second root is always a repelling fixed point of the dynamic system defined by $Q_c(z) = z^2 + c$. [30]

**root-finding route to chaos**: Describes the historical evolution of the Julia and Mandelbrot theory, which began in antiquity with root-finding algorithms, and evolved into part of the branch of modern mathematics known as dynamic system theory. [17]

**San Marco**: Meaningful for those who have been to Venice (or seen pictures thereof), this is how Mandelbrot described one of the quadratic Julia sets he discovered. [32]

**satellite**: Refers to one of the miniature Mandelbrot subsets that are found within the Mandelbrot set. Also known as “baby Mandelbrots”, they are not exact copies. The terminology “satellite” sets up a conflict with the connectedness of the Mandelbrot set. [45,249,250,259]

**Seahorse Valley**: Describes that part of the Mandelbrot set that is found between the main cardioid and the 2-cycle disk. [53]

**seed value**: The initial term in a sequence that is generated recursively. [1]

**self-similarity**: In its strict geometric sense, this property is best illustrated by the Koch snowflake: Any side is similar to vanishingly small pieces thereof. Julia sets have an analogous property, except that complex functions (which are *conformal*) take the place of similarity transformations. [25,27,66,250]

**simple root**: If $f(z)$ is a polynomial function, then $a$ is a simple root of $f$ if $(z - a)^2$ is not a factor of $f(z)$. An equivalent formulation is that $f(a) = 0$ and $f'(a) \neq 0$. [6]

**slow convergence**: Even though a sequence is known to approach a given limit, it can still take a unbelievably long time to get there. The slowness can mean a diminishing acceleration toward an indifferent limit, or a very gradual acceleration (as from a great distance) toward an attracting limit. [38,220,243]

**snowflake curve**: Now regarded as one of the earliest examples of a self-similar fractal, this pathological example was invented in 1904 by the Swedish mathematician Helge von Koch for a different purpose. It is a continuous simple closed curve that has infinite length and that is non-differentiable at all its points. [31]

**spirals**: A conspicuous motif in complex fractals. [27,28,31,49,240,243]
stereographic projection: Adds an “infinite” point to the complex-number system, by wrapping the complex plane around a sphere. The result is called the “complex sphere”. Complex numbers with large magnitudes are thus regarded as being close to infinity, as well as close to each other. This model distorts the sense of distance, but it does preserve angular relationships. [18]

superattracting fixed point: Given a dynamic system $z_n = f(z_{n-1})$, a fixed point $p$ is called superattracting if $|f'(p)| = 0$. This means that all orbits that originate suitably close to $p$ will converge at least quadratically to $p$. [9,17,21]

superattracting cycle: Given a dynamic system $z_n = f(z_{n-1})$, a $k$-cycle $\{p_1, p_2, \ldots, p_k\}$ is called attracting if $|f'(p_1)||f'(p_2)| \cdots |f'(p_k)| = 0$. This is equivalent to saying that one of the members of the cycle is an attracting fixed point for $f^k$, and it means that all orbits that originate suitably close to $p_1$ will rapidly approach $\{p_1, p_2, \ldots, p_k\}$ as a limiting cycle. For quadratic examples $Q_c(z) = z^2 + c$, a superattracting cycle must include $0$. [30,234,240,246]

tangent-line bifurcation: The name given to the sudden appearance of an indifferent $k$-cycle, which then splits into two $k$-cycles, one attracting and one repelling. In the quadratic world catalogued by the Mandelbrot set, this occurs when $c$ moves into a cardioid through its cusp point. The terminology stems from the corresponding web diagram, which shows how a slight change in the value of a parameter can cause a graph $y = f_c^k(x)$ to become tangent to the line $y = x$. [165]

trifurcation: Splitting into three. In this book, the terminology refers to the birth of an attracting $3k$-cycle from a formerly attracting $k$-cycle. Also called period-tripling. [40,44]

web diagram: A device for visualizing the long-term behavior of sequences generated by a real-valued dynamic system. [38,68,212,213,220,225,248,252,259]
Solutions
1. The function \( f(x) = \frac{1}{2} \left( x + \frac{3}{x} \right) \), when applied to the seed value \( x_0 = 2 \), generates the following sequence of rational approximations to \( \sqrt{3} \):

\[
\begin{align*}
x_0 &= \frac{2}{1} \\
x_1 &= \frac{7}{4} \\
x_2 &= \frac{97}{56} \\
x_3 &= \frac{18817}{10864} \\
x_4 &= \frac{708158977}{408855776} \\
x_5 &= \frac{1002978273411373057}{579069776145402304} \\
x_6 &= \frac{2011930833870518011412817828051050497}{1161588808526051807570761628582646656}
\end{align*}
\]

The corresponding decimal values are

\[
\begin{align*}
x_0 &= 2.0 \\
x_1 &= 1.75 \\
x_2 &= 1.73214\ldots \\
x_3 &= 1.732050810\ldots \\
x_4 &= 1.73205080756887729525\ldots \\
x_5 &= 1.7320508075688772935274463415058723678\ldots \\
x_6 &= 1.732050807568877293527446341505872366942805253810380628055806979451933017 \\
\text{and} \\
\sqrt{3} &= 1.732050807568877293527446341505872366942805253810380628055806979451933016
\end{align*}
\]

It is customary to say that two numbers agree to \( k \)-place accuracy if the absolute value of their difference is smaller than 0.5 times \( 10^{-k} \). Thus \( x_1 \) is accurate to 1 place, \( x_2 \) to 3 places, \( x_3 \) to 8 places, \( x_4 \) to 17 places, \( x_5 \) to 35 places, and \( x_6 \) to 71 places.

The rate of convergence that is illustrated here is called \textit{quadratic}. Roughly speaking, this means that the number of correct decimal places \textit{doubles} with each application of the algorithm. The actual definition is that there exists a positive constant \( C \) with the property that

\[
|x_n - x_\infty| \leq C|x_{n-1} - x_\infty|^2
\]

holds for all positive \( n \), where \( x_\infty \) is the limiting value of the sequence \( x_0, x_1, x_2, \ldots \). In the current example, \( C = 0.3 \) works.
2. Calculate 4(AM)^2 - 4(GM)^2 = (p+q)^2 - 4pq = p^2 - 2pq + q^2 = (p-q)^2, which is never negative. Equality occurs exactly when p = q. In the Babylonian algorithm, the target root is always the geometric mean of the bracketing values \( x_n \) and \( 3/x_n \). The algorithm proceeds by approximating this with the arithmetic mean of \( x_n \) and \( 3/x_n \), which is always slightly larger than the target root.

3. The inequalities \( 0 < x < y < 3/x \) imply that \( 0 < x/3 < 1/y < 1/x \), hence that \( x < 3/y < 3/x \). The bracketing values move closer together with each application of the Babylonian algorithm.

4. If a product of two different numbers is 3, one of the numbers must be smaller than \( \sqrt{3} \) and the other number must be greater than \( \sqrt{3} \). Thus the target root is trapped between the bracketing values \( x \) and \( 3/x \).

5. A simple program for the TI-83 calculator:

   ```
   : Y_1 STO X
   : DISP X
   ```

   Enter the function to be iterated as \( Y_1 \), then store a seed value in the \( X \)-register. Each time the above program is executed (once the program is loaded, just press ENTER), it will calculate the next value in the sequence, store it in the \( X \)-register, and display it.

6. The algorithm is defined by the function \( f(x) = \frac{1}{2} \left( x - \frac{1}{x} \right) \). The seed value \( x_0 = 2 \) produces the sequence

   \[ 2.000 \, , \, 0.750 \, , \, -0.292 \, , \, 1.568 \, , \, 0.465 \, , \, -0.842 \, , \, 0.173 \, , \, \ldots \]

   which seems to be going nowhere. The seed value \( x_0 = 1/\sqrt{3} \) produces the sequence

   \[ 0.577 \, , \, -0.577 \, , \, 0.577 \, , \, -0.577 \, , \, 0.577 \, , \, -0.577 \, , \, \ldots \]

   which apparently oscillates forever.

7. One function that works is \( f(x) = \frac{1}{2} \left( x + \frac{3}{x^2} \right) \), although it works only slowly. How could you improve \( f \)? Well, notice that \( \sqrt[3]{3} \) is the geometric mean of \( x \), \( x \), and \( 3/x^2 \) (in general, the geometric mean of \( three \) positive quantities \( p \), \( q \), and \( r \) is \( \sqrt[3]{pqr} \)). Should your function be based on the arithmetic mean of only \( two \) quantities?

8. The expression simplifies to \( \cot 2\theta \). Now notice that \( f(\cot \theta) = \cot 2\theta \) applies in item 6, so that it is possible to calculate \( x_{20} = -0.468 \) from \( x_0 = 2 \) without calculating \( x_{19} \) first.

9. The seed value \( x_0 = 1 \) produces

   \[ 1.000 \, , \, 1.316 \, , \, 1.410 \, , \, 1.434 \, , \, 1.440 \, , \, 1.442 \, , \, 1.442 \, , \, \ldots \]

   a sequence that approaches the real cube root of 3.
Dynamic Fractal Solutions

1. (a) $5 + 5i$  (b) $1 + 3i$  (c) $2 + 11i$  (d) $2 + i$

2. $(a + bi)(a - bi) = a^2 + b^2$ and $(a + bi) + (a - bi) = 2a$.

3. The real number system is included as part of the complex number system — just consider numbers of the form $a + 0i$ — so some complex numbers are real.

4. $1 + i = \sqrt{2}\text{cis}(45.000), 2 + i = \sqrt{5}\text{cis}(26.565),$ and $1 + 3i = \sqrt{10}\text{cis}(71.565)$. The magnitude of the product equals the product of the magnitudes; the argument of the product (that means the angle) equals the sum of the arguments. These are not coincidences.

5. $z$ is on (a) the circle of radius 3, centered at the origin; (b) the circle of radius 3, centered at $2 = (2, 0)$; (c) the line $x = 2$; (d) the perpendicular bisector of the segment joining $i$ to 2; (e) the imaginary axis $x = 0$; (f) the line $x + 2y = 0$ ($z = 2 - i$, for instance); (g) either coordinate axis; (h) the complex plane.

6. (a) $x^2 - y^2 + 2xyi$  (b) $\frac{3x + 4y}{x^2 + y^2} + \frac{4x - 3y}{x^2 + y^2}i$  (c) $4096 + 4096i$  (d) $-1$

7. Because

$$e^z = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \ldots = \sum_{n=0}^{\infty} \frac{z^n}{n!},$$

it follows that

$$e^{i\theta} = 1 + i\theta - \frac{1}{2}\theta^2 - \frac{1}{6}i\theta^3 + \frac{1}{24}\theta^4 + \ldots$$

$$= \left\{1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4 + \ldots\right\} + i\left\{\theta - \frac{1}{6}\theta^3 + \frac{1}{120}\theta^5 - \ldots\right\}$$

$$= \cos \theta + i\sin \theta.$$  

8. Apply rules of exponents to find that

$$\text{cis}(\alpha + \beta) = e^{\alpha + \beta} = e^\alpha e^\beta = \text{cis}(\alpha)\text{cis}(\beta).$$

Or, apply trigonometric identities to find that

$$\text{cis}(\alpha + \beta) = \cos(\alpha + \beta) + i\sin(\alpha + \beta)$$

$$= \cos \alpha \cos \beta - \sin \alpha \sin \beta + i(\sin \alpha \cos \beta + \cos \alpha \sin \beta)$$

$$= (\cos \alpha + i\sin \alpha)(\cos \beta + i\sin \beta)$$

$$= \text{cis}(\alpha)\text{cis}(\beta).$$

9. (a) $\log(-1) = (2n + 1)i\pi$; (b) $\log i = (2n + \frac{1}{2})i\pi$, so $i^i = e^{\log i} = e^{-(4n+1)\pi/2}$, which gives $i^i$ the values $e^{-\pi/2} = 0.2079$, or $e^{3\pi/2} = 111.32$, or $e^{-5\pi/2} = 0.000382$, or ...
1. Notice that \( \frac{a}{b} < \frac{c}{d} \implies ad < bc \implies ab + ad < ab + bc \implies \frac{a}{b} < \frac{a+c}{b+d} \). The other inequality is treated similarly.

2. The data below was produced by applying the function \( f(x) = \frac{x + 3}{x + 1} \) twenty-eight times to the seed value \( x_0 = 2 \). Each entry in the error column is — in magnitude — about 27% of the entry just above it. This is an example of linear convergence.

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<th>( x_n - \sqrt{3} )</th>
</tr>
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<td>-0.000000000001</td>
</tr>
<tr>
<td>1.7320508075688773</td>
<td>0.000000000000</td>
</tr>
</tbody>
</table>

Although this root-finding function works, it is significantly slower than the Babylonian function, which would have established well over three hundred million accurate decimal places by the time \( x_{28} \) was reached.
2. Notice the special role played by $-\sqrt{3}$. Although it is also a fixed point of $f$, it never appears as a sequential limit, no matter what $x_0$ is. In fact, even if you set $x_0 = -\sqrt{3}$, the resulting sequence (which should be constant) will approach $\sqrt{3}$ when calculated by machine. Why?

3. It is the continuity of $f$ at $p$ that should lead you to expect that

$$f(p) = f \left( \lim_{n \to \infty} x_n \right) = \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} x_{n+1} = p.$$

4. The sequence $1, \cos(1), \cos(\cos(1)), \ldots$ converges to 0.7390851332.

5. No! For example, take $x = 1$: $y = 2$ is between 1 and 3, but $3/y^2 = 3/4$ is not between 1 and 3. This means that the nested-interval explanation of convergence fails for this cube-root finder. Nevertheless, for almost every seed value, the process does converge to the real cube root of 3.

6. The fixed points are found by solving $x = \frac{1}{3} \left( 2x + \frac{3}{x^2} \right)$, which leads only to $x = \sqrt[3]{3}$. Let $m$ stand for $\sqrt[3]{3}$, so that $f(x) = \frac{1}{3} \left( 2x + \frac{m^3}{x^2} \right)$. The approximation error can be written in the form

$$f(x) - m = \frac{(2x + m)(x - m)^2}{3x^2}.$$

This shows that $m \leq f(x)$ whenever $x$ is positive. The difference $x - f(x)$ can be written in the form $\frac{x^3 - m^3}{3x^2}$, which shows that $f(x) < x$ whenever $m < x$. It follows that, given a positive seed value $x_0$ other than $m$, the ensuing sequence $x_1, x_2, \ldots$ decreases towards $m$. It has to converge to some limiting value, which can only be a fixed point of $f$. Thus the limit is $m$. Next, what if $x < 0$? The expression for $x - f(x)$ shows that $x < f(x)$, suggesting that the sequence $x_1, x_2, \ldots$ will drift in the positive direction — until it moves to the right of $m$, at which time it will begin its descending approach to $m$. There are some troublesome nonpositive $x$-values, however. The most obvious one is 0, for $f(0)$ is undefined. The solution to $f(x) = 0$, which is $-\sqrt{1.5}$, is therefore also troublesome, as is the solution to $f(x) = -\sqrt[3]{1.5}$, etc. In other words, there are infinitely many negative seed values $x_0$ for which the sequence $x_0, x_1, x_2, \ldots$ is eventually undefined.

7. Write $f(x) - m = (x - m)^2/(2x)$. Without loss of generality, assume that $x_0$ and $m$ are both positive. It follows that $x_{n+1} = f(x_n)$ is greater than $m$ for all nonnegative $n$, and that

$$|x_{n+1} - m| = \frac{1}{2x_n} |x_n - m|^2 \leq \frac{1}{2m} |x_n - m|^2$$

holds for positive $n$. This satisfies the definition of quadratic convergence.
Dynamic Fractal Solutions

8. Unless your calculator has been designed to understand the complex variable \( z \), it is necessary to write

\[
  f(z) = \frac{1}{2} \left( z - \frac{1}{z} \right) = \frac{1}{2} \left( x + yi - \frac{1}{x + yi} \right) = \frac{1}{2} \left( x + yi - \frac{x - yi}{x^2 + y^2} \right),
\]

and then isolate the real and imaginary parts

\[
  \frac{x^3 + xy^2 - x}{2(x^2 + y^2)} \quad \text{and} \quad \frac{x^2y + y^3 + y}{2(x^2 + y^2)}.
\]

A simple TI-83 program to examine convergence data:

: \( Y_1 \) STO \( T \)
: \( Y_2 \) STO \( Y \)
: \( T \) STO \( X \)
: DISP ”\( X = \)” , \( X \)
: DISP ”\( Y = \)” , \( Y \)

Enter the real and imaginary parts as functions \( Y_1 \) and \( Y_2 \), then store a seed value \( x_0 + y_0i \) in registers \( X \) and \( Y \). Each time the above program is executed, it will calculate the next complex number in the sequence, move it to the \( X \)- and \( Y \)-registers (notice the temporary use of the \( T \)-register), and display it. The data below was produced by applying \( f \) eight times to the seed value \( 1 + 0.25i \).

| \( x_n \)  | \( y_n \)  | \( |z_n - i| \) |
|----------|----------|-----------|
| 1.0000000000000000 | 0.2500000000000000 | 1.2500000000000000 |
| 0.0294117647058824 | 0.2426470588235294 | 0.7579238282385405 |
| -0.2314479638009050 | 2.1520927601809955 | 1.1751110109300524 |
| -0.0910234017356160 | 1.3057219359780503 | 0.3189845792537477 |
| -0.0189463069861897 | 1.0339389860340656 | 0.0388692336100860 |
| -0.0006146788325344 | 1.0003946964102081 | 0.0007304897832257 |
| -0.0000002425839062 | 0.9999999891704498 | 0.0000002667023449 |
| 0.0000000000000026 | 0.9999999999999976 | 0.0000000000000356 |
| -0.0000000000000000 | 1.0000000000000000 | 0.0000000000000000 |

The third column tables the distance from \( z_n \) to \( i \). Notice that, although the root-finding function did its job, the sequence \( z_0 \), \( z_1 \), \( z_2 \), \ldots did not at first approach \( i \) in a steady fashion. Once the sequence came close enough to its limit, however, the convergence behaved quadratically.
9. This function also produces sequences that approach the square roots of 3. What is remarkable about this example is that the rate of convergence is cubic. Roughly speaking, this means that the accuracy triples with each application of the algorithm. (What is the precise definition, however?) The data below was produced by applying the function $f(x) = \frac{x^3 + 9x}{3x^2 + 3}$ three times to the seed value $x_0 = 2$.

<table>
<thead>
<tr>
<th>$x_n$</th>
<th>$x_n - \sqrt{3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.0000000000000000</td>
<td>0.2679491924311227</td>
</tr>
<tr>
<td>1.7333333333333333</td>
<td>0.0012825257644560</td>
</tr>
<tr>
<td>1.7320508077444814</td>
<td>0.0000000001756041</td>
</tr>
<tr>
<td>1.7320508075688773</td>
<td>0.0000000000000000</td>
</tr>
</tbody>
</table>

This root finder is a special case of Hutton’s method, which employs the function

$$f(x) = \frac{x^3 + 3Ax}{3x^2 + A}$$

to recursively generate sequences that converge to $\pm \sqrt{A}$.

10. Lacking any additional knowledge about $f$, the best possible answer you can give is that

$$f(2.3) = f(2) + 0.3f'(2) = 7 + (0.3)(-0.5) = 6.85.$$  

In effect, you are simply assuming that $f$ is a linear function.
Dynamic Fractal Solutions

1. Consider the Babylonian \( f(x) = \frac{1}{2}(x + 3x^{-1}) \), for which there are three possibilities:
   If \( x_0 \) is positive, then \( x_n \to \sqrt{3} \); if \( x_0 \) is negative, then \( x_n \to -\sqrt{3} \); if \( x_0 = 0 \), then the sequence \( \{x_n\} \) is not defined.

2. The two sequences are
   
   \[
   3.0, 2.4, 2.16, 2.064, 2.0256, 2.01024, \ldots
   \]

   and

   \[
   -5.0, -0.8, 0.88, 1.552, 1.8208, 1.92832, \ldots
   \]

   in each of which the distance to the limiting value of 2 is diminished according to the rule \( x_{n+1} - 2 = 0.4(x_n - 2) \).

3. Check by using DeMoivre’s Theorem to calculate

   \[
   \text{cis}(144) \times 5 = \text{cis}(5 \times 144) = \text{cis}(720) = 1.
   \]

   The other solutions are \( \text{cis}(72) \), \( \text{cis}(216) \), \( \text{cis}(288) \), and \( \text{cis}(360) = \text{cis}(0) \). These five unit complex numbers are evenly spaced around the unit circle, thereby defining a regular pentagon.

4. Given a seed value \( x_0 \) that is between \(-1\) and 1, the sequence
   
   \[
   x_0, x_1 = x_0^2, x_2 = x_0^4, x_3 = x_0^8, \ldots, x_n = x_0^{2^n}, \ldots
   \]

   approaches 0 quadratically. If \( 1 < |x_0| \), there is no finite limit. If \( |x_0| = 1 \), the sequence is eventually constant.

5. Given a seed value \( z_0 \) whose magnitude is less than 1, the sequence
   
   \[
   z_0, z_1 = z_0^2, z_2 = z_0^4, z_3 = z_0^8, \ldots, z_n = z_0^{2^n}, \ldots
   \]

   approaches 0 quadratically. If \( 1 < |z_0| \), there is no finite limit. If \( |z_0| = 1 \), there are many possibilities.

6. Cubic convergence means that there exists a positive constant \( C \) with the property that

   \[
   |x_n - x_\infty| \leq C|x_{n-1} - x_\infty|^3
   \]

   holds for all positive \( n \), where \( x_\infty \) is the limiting value of the sequence \( x_0, x_1, x_2, \ldots \).

7. Because \( e^{z+2\pi i} = e^z \), one possible period is \( 2\pi i \). Because \( 1 = e^z = e^x e^{yi} = e^x \text{cis} y \) can occur only if \( x = 0 \) and \( y = 2n\pi \), you see that \( 2\pi i \) is the fundamental period.
8. The simplest way is to substitute into the Maclaurin series
\[ \cos z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!} \quad \text{and} \quad \cosh z = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!}, \]
or into the Maclaurin series
\[ \sin z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!} \quad \text{and} \quad \sinh z = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!}. \]

9. Apply the familiar identities
\[
\begin{align*}
\cos(x + yi) &= \cos x \cos yi - \sin x \sin yi \\
\sin(x + yi) &= \sin x \cos yi + \cos x \sin yi \\
\cosh(x + yi) &= \cosh x \cosh yi + \sinh x \sinh yi \\
\sinh(x + yi) &= \sinh x \cosh yi + \cosh x \sinh yi
\end{align*}
\]
and use the preceding item.

10. This is linear convergence, with \(C = \frac{1}{7}\).

11. This is quadratic convergence, with \(C = 1\).

12. Define \(x_n = \left(\frac{19}{94}\right)^{3n}\), a sequence that approaches 0 cubically.

13. Let \(f(x) = \frac{1}{3}(5 - x^2)\), and apply this function repeatedly to any seed value \(x_0\) chosen between \(-4\) and 1.5. The resulting sequence will approach 1.19258240..., a solution that is usually delivered by the quadratic formula as \(\frac{1}{2}(-3 + \sqrt{29})\).

Another way to rewrite the given quadratic equation, by solving for \(x\), is \(x = (5 - 3x)/x\), which suggests that the function \(g(x) = \frac{1}{x}(5 - 3x)\) could be used recursively to solve the equation. Indeed, for almost any seed value \(x_0\), the resulting sequence will approach \(-4.19258240\ldots\), better known as \(\frac{1}{2}(-3 - \sqrt{29})\). By the way, what values of \(x_0\) do not initiate convergent sequences?

In the two examples above, convergence is only linear, and neither dynamic system detects both solutions to the quadratic equation.

A third way of rewriting the equation is \(x = (x^2 + 5)/(2x + 3)\), which suggests trying the dynamic system defined by \(h(x) = \frac{x^2 + 5}{2x + 3}\). With one exception, every real seed value \(x_0\) initiates a sequence that converges quadratically to one of the two solutions, and both solutions are detected by \(h\).
1. Lacking any additional knowledge about $f$, the best possible answer you can give is that
\[ f(2.3) = f(2) + 0.3f'(2) = 7 + (0.3)\frac{i}{2} = 7 + 0.15i. \]
In effect, you are simply assuming that $f$ is a linear function.

2. The triangle formed by $\frac{1}{2}iz_1$, $\frac{1}{2}iz_2$, and $\frac{1}{2}iz_3$ is exactly half the size of the triangle formed by $z_1$, $z_2$, and $z_3$. Multiplication by $\frac{1}{2}i$ has given the figure a 90-degree, counterclockwise turn about the origin, and shrunk the figure by half.

3. The configuration is given a 90-degree, counterclockwise turn about the origin, and shrunk by half.

4. This is linear convergence, with $C = \frac{1}{10}$.

5. Unless your calculator has been designed to understand the complex variable $z$, it is necessary to write
\[ f(z) = \frac{1}{2} \left( z + \frac{3+4i}{z} \right) = \frac{1}{2} \left( x + yi + \frac{3+4i}{x + yi} \right) = \frac{1}{2} \left( x + yi + \frac{(3+4i)(x - yi)}{x^2 + y^2} \right), \]
and then store the real and imaginary parts
\[ \frac{1}{2} \left( x + \frac{3x + 4y}{x^2 + y^2} \right) \quad \text{and} \quad \frac{1}{2} \left( y + \frac{4x - 3y}{x^2 + y^2} \right) \]
as separate functions ($Y_1$ and $Y_2$, say) in your calculator. Here are some of the results of running the program shown on page 206:

<table>
<thead>
<tr>
<th>$x_n$</th>
<th>$y_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0000000000000000</td>
<td>3.0000000000000000</td>
</tr>
<tr>
<td>1.2500000000000000</td>
<td>1.2500000000000000</td>
</tr>
<tr>
<td>2.0250000000000000</td>
<td>0.82500000000000000</td>
</tr>
<tr>
<td>1.9928921568627451</td>
<td>1.0007352941176471</td>
</tr>
<tr>
<td>2.0000089628275817</td>
<td>0.9999928767981272</td>
</tr>
<tr>
<td>1.9999999999931498</td>
<td>0.9999999999715032</td>
</tr>
<tr>
<td>2.00000000000000000</td>
<td>1.00000000000000000</td>
</tr>
</tbody>
</table>

This table shows that the seed value $1 + 3i$ leads to the root $2 + i$. In a similar fashion, it is found that the seed $-1 + 3i$ leads to the same root. On the other hand, the seed $-1 + i$ leads to the root $-2 - i$. Turn the page.
The table below shows that not all seed values lead to a root.

<table>
<thead>
<tr>
<th>$x_n$</th>
<th>$y_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.0000000000000000</td>
<td>-4.0000000000000000</td>
</tr>
<tr>
<td>0.7500000000000000</td>
<td>-1.5000000000000000</td>
</tr>
<tr>
<td>-0.2916666666666667</td>
<td>0.5833333333333333</td>
</tr>
<tr>
<td>1.5684523805238100</td>
<td>-3.1369047619047619</td>
</tr>
<tr>
<td>0.4654406117285624</td>
<td>-0.9308812234571248</td>
</tr>
<tr>
<td>-0.8415306026309837</td>
<td>1.6830612052619674</td>
</tr>
<tr>
<td>0.1733901559391661</td>
<td>-0.3467803118783322</td>
</tr>
<tr>
<td>-2.7969749740685780</td>
<td>5.5939499481371559</td>
</tr>
<tr>
<td>-1.2197229272382167</td>
<td>2.4394458544764334</td>
</tr>
<tr>
<td>-0.1999322951613010</td>
<td>0.3998645990322601</td>
</tr>
<tr>
<td>2.4008039284702922</td>
<td>-4.8017607856940584</td>
</tr>
<tr>
<td>0.9921832580563824</td>
<td>-1.9843665161127647</td>
</tr>
<tr>
<td>-0.0078475335947600</td>
<td>0.0156950667189521</td>
</tr>
<tr>
<td>63.7103641625637076</td>
<td>-127.4207283251274170</td>
</tr>
<tr>
<td>31.8473340646118471</td>
<td>-63.6946681292236950</td>
</tr>
<tr>
<td>15.9079671310524748</td>
<td>-31.8159342621049500</td>
</tr>
<tr>
<td>7.9225527739058552</td>
<td>-15.8451055478117106</td>
</tr>
<tr>
<td>3.8981654157491635</td>
<td>-7.7963308314918327</td>
</tr>
<tr>
<td>1.8208172427953751</td>
<td>-3.641634855907502</td>
</tr>
<tr>
<td>0.6358066524310577</td>
<td>-1.2716133048621155</td>
</tr>
<tr>
<td>-0.4684992665981982</td>
<td>0.9369985331963964</td>
</tr>
<tr>
<td>0.8329878965065087</td>
<td>-1.6659757930130177</td>
</tr>
<tr>
<td>-0.1837548694029973</td>
<td>0.3675097388059944</td>
</tr>
<tr>
<td>2.6291388933253673</td>
<td>-5.2582778665073822</td>
</tr>
<tr>
<td>1.1243931112589676</td>
<td>-2.2487862225179372</td>
</tr>
<tr>
<td>0.117512232609256</td>
<td>-0.235024465218529</td>
</tr>
<tr>
<td>-4.1961204120635355</td>
<td>8.3922408241270051</td>
</tr>
<tr>
<td>-1.9789025196692561</td>
<td>3.9578050393384773</td>
</tr>
<tr>
<td>-0.7367859592297415</td>
<td>1.4735719184594611</td>
</tr>
<tr>
<td>0.3102301588101818</td>
<td>-0.6204603176203947</td>
</tr>
<tr>
<td>-1.4565915840936611</td>
<td>2.9131830768185552</td>
</tr>
<tr>
<td>-0.3850286371259039</td>
<td>0.7700572742516775</td>
</tr>
<tr>
<td>1.1060903871345637</td>
<td>-2.212180774269320</td>
</tr>
<tr>
<td>0.1010025704546271</td>
<td>-0.2020051409097126</td>
</tr>
<tr>
<td>-4.8998677771582092</td>
<td>9.7997355542937198</td>
</tr>
<tr>
<td>-2.3478903186830234</td>
<td>4.6957806373542247</td>
</tr>
<tr>
<td>-0.9609880224508670</td>
<td>1.9219760448947507</td>
</tr>
</tbody>
</table>

If the entries in the table appear random to you, look again — there are some significant patterns to be seen.
6. The ratios approach zero when the root-finding process is defined by $g$, because

$$\frac{|x_n - \sqrt[3]{5}|}{|x_{n-1} - \sqrt[3]{5}|} \leq C \frac{|x_{n-1} - \sqrt[3]{5}|^2}{|x_{n-1} - \sqrt[3]{5}|} = C |x_{n-1} - \sqrt[3]{5}| \to 0.$$ 

Another way to see this is to realize that the limiting value of

$$\frac{x_n - \sqrt[3]{5}}{x_{n-1} - \sqrt[3]{5}}$$

is simply $g'(\sqrt[3]{5})$ (just recognize the definition of $g'$). Thus calculate

$$g'(x) = \frac{1}{3} \left( \frac{2}{x^3} \right)$$

and then substitute to find that $g'(\sqrt[3]{5}) = 0$. The same principle applies to the $f$-process, so the limiting value is $f'(\sqrt[3]{5}) = -\frac{1}{2}$.

7. The figure at right shows the graphs of $y = x$ and

$$f(x) = \frac{1}{2} \left( x + \frac{5}{x^2} \right).$$

The lone intersection is $\left( \sqrt[3]{5}, \sqrt[3]{5} \right)$ — the target point of your search. As the discussion for the preceding item concluded, the curve crosses the line $y = x$ with slope $-0.5$, which is why the errors change sign and are eventually diminished by roughly 50 per cent with each application of the algorithm.

The dotted line traces the first three applications of $f$ to the seed value $x_0 = -1.3$, which is represented by the point $(-1.3, -1.3)$ on the line $y = x$. There are two dotted segments for each application of $f$ — the first (vertical) one connects $(x_n, x_n)$ to $(x_n, x_{n+1})$, while the second (horizontal) one connects $(x_n, x_{n+1})$ to $(x_{n+1}, x_{n+1})$. These two segments correspond to evaluating $f(x_n)$ and then converting this $y$-value back into an $x$-value, respectively. Notice how the sequence

$$-1.3, 0.8293, 4.0498, 2.1774, 1.6160, \ldots$$

is being drawn (slowly) toward the cube root.
page 5

8. The figure at right shows the graphs of \( y = x \) and
\[
f(x) = \frac{x + 3}{x + 1}.
\]
The intersections are \((-\sqrt{3}, -\sqrt{3})\) and \((\sqrt{3}, \sqrt{3})\), which are the targets of your search. As the discussion in item 6 on page 212 indicated, the slopes of the curve at the fixed points are of interest. For this example,
\[
f'(x) = \frac{-2}{(x + 1)^2},
\]
from which you can calculate
\[
f'(-\sqrt{3}) = -2 - \sqrt{3} = -3.732 \quad \text{and} \quad f'(\sqrt{3}) = -2 + \sqrt{3} = -0.268.
\]
The second slope explains why small errors that occur when you try to approximate \( \sqrt{3} \) are diminished to roughly 27% by each application of this algorithm (see the table of errors on page 4). It also explains why the sign of the error alternates.

The dotted line traces the first four applications of \( f \) to the seed value \( x_0 = -1.4 \), which is represented by the point \((-1.4, -1.4)\) on the line \( y = x \). There are two dotted segments for each application of \( f \) — the first (vertical) one connects \((x_n, x_n)\) to \((x_n, x_{n+1})\), while the second (horizontal) one connects \((x_n, x_{n+1})\) to \((x_{n+1}, x_{n+1})\). These two segments correspond to evaluating \( f(x_n) \) and then converting this \( y \)-value back into an \( x \)-value, respectively. Notice how the sequence
\[-1.4, -4.0, 0.3333, 2.5, 1.5714, \ldots\]
is being drawn (slowly) toward the positive root.

The figure and the slope \( f'(-\sqrt{3}) \) both make it clear that no sequence is going to approach \(-\sqrt{3}\) as a limiting value. Even if you seed a sequence with the improbable value \( x_0 = -\sqrt{3} \), your calculator will not be able to deliver the perfect accuracy that this example demands, and the resulting sequence will still approach \( \sqrt{3} \). Try it.
Dynamic Fractal Solutions

1. An equation for the tangent line is

\[
\frac{y - E(a)}{x - a} = E'(a).
\]

To obtain the x-intercept, just set \( y = 0 \) and solve for \( x \).

2. Some of the functions \( N \) below should look familiar:

\[
\begin{align*}
(a) N(x) &= x - \frac{x^2 - 3}{2} = \frac{1}{2} \left( x + \frac{3}{x} \right) \\
(b) N(x) &= x - \frac{x^3 - 5}{3x^2} = \frac{1}{3} \left( 2x + \frac{5}{x^2} \right) \\
(c) N(x) &= x - \frac{x^n - A}{nx^{n-1}} = \frac{1}{n} \left( (n-1)x + \frac{A}{x^{n-1}} \right) \\
(d) N(x) &= x - \frac{x - \cos x}{1 + \sin x} = \frac{x \sin x + \cos x}{1 + \sin x} \\
(e) N(x) &= x - \frac{x^2 + 3x - 5}{2x + 3} = \frac{x^2 + 5}{2x + 3}
\end{align*}
\]

Each of these five dynamic systems converges quadratically to its fixed points, which represent all of the solutions of the associated equation \( E(x) = 0 \).

3. Although it is in direct conflict with the practice of writing \( \sin^2 x \) when \( (\sin x)^2 \) is meant, the customary notation for \( f(f(f(f(f(x)))))) \) is \( f^6(x) \). Now, how does this relate to the notation \( f^{-1}(x) \) for the function that is inverse to \( f \)?

4. Because the magnitude of \( f'(\sqrt{3}) \) is greater than 1, errors will grow exponentially as \( n \) increases. Notice that \( x_0 - \sqrt{3} \approx -0.00005080757 \), hence \( x_1 - \sqrt{3} \approx -0.0000812921 \), \( x_2 - \sqrt{3} \approx -0.0001300674 \), and — finally — \( x_{21} - \sqrt{3} \approx -0.00005080757(1.6)^{21} \), hence that \( x_{21} \approx \sqrt{3} - 0.9827613096 \approx 0.7492894980 \).

5. Assuming that \( E(m) = 0 \) and \( E'(m) \neq 0 \), calculate

\[
N'(m) = 1 - \frac{E'(m)E'(m) - E(m)E''(m)}{E'(m)E'(m)} = \frac{E(m)E''(m)}{E'(m)E'(m)} = 0
\]

6. This illustrates how a multiple root can slow down Newton’s method. The equation factors as \( (x - 1)^3 = 0 \), so \( N(x) = x - \frac{1}{3}(x - 1) = \frac{1}{3} \left( 2x + 1 \right) \). Convergence is only linear, because \( N'(1) = \frac{2}{3} \).
1. Seed points that are near the line \( y = -2x \) initiate sequences that will take a long time (i.e., many steps) to converge. For instance consider the table on page 211. With the exception of the first couple of points in the list, these points are not really on the line \( y = -2x \), they are only close. Here are a few more terms of the same sequence:

<table>
<thead>
<tr>
<th>( x_n )</th>
<th>( y_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.9609880224508670</td>
<td>1.9219760448947507</td>
</tr>
<tr>
<td>0.039803867381854</td>
<td>-0.0796076734836434</td>
</tr>
<tr>
<td>-12.5417012077016624</td>
<td>25.0834024131045575</td>
</tr>
<tr>
<td>-6.2309836040140819</td>
<td>12.4619672068714730</td>
</tr>
<tr>
<td>-3.0352476492736116</td>
<td>6.0704952979539817</td>
</tr>
<tr>
<td>-1.3528926205389594</td>
<td>2.7057852407491011</td>
</tr>
<tr>
<td>-0.3068678290143951</td>
<td>0.6137356577745562</td>
</tr>
<tr>
<td>1.4759320623888200</td>
<td>-2.9518641262546552</td>
</tr>
<tr>
<td>0.3991970507369072</td>
<td>-0.7983941025513396</td>
</tr>
<tr>
<td>-1.0529157381437975</td>
<td>2.1058314723680065</td>
</tr>
<tr>
<td>-0.0515860613036359</td>
<td>0.1031721188797178</td>
</tr>
<tr>
<td>9.6667467938085378</td>
<td>-19.3334960898536022</td>
</tr>
<tr>
<td>4.7816501425274344</td>
<td>-9.5633006399305771</td>
</tr>
<tr>
<td>2.2862586590218306</td>
<td>-4.5725175032420324</td>
</tr>
<tr>
<td>0.9244314153326819</td>
<td>-1.8488629409801716</td>
</tr>
<tr>
<td>-0.0786573000083927</td>
<td>0.1573144803156034</td>
</tr>
<tr>
<td>6.3173604296669634</td>
<td>-12.6347305928463852</td>
</tr>
<tr>
<td>3.0795332347669793</td>
<td>-6.1590714582362402</td>
</tr>
<tr>
<td>1.3774043495063672</td>
<td>-2.7548114563829450</td>
</tr>
<tr>
<td>0.3257005622065202</td>
<td>-0.651403297752686</td>
</tr>
<tr>
<td>-1.3723021196432316</td>
<td>2.7445932632722623</td>
</tr>
<tr>
<td>-0.3217998044846410</td>
<td>0.6435912067714328</td>
</tr>
<tr>
<td>1.3928611410018479</td>
<td>-2.7857670526747979</td>
</tr>
<tr>
<td>0.3374572420203483</td>
<td>-0.674984075426116</td>
</tr>
<tr>
<td>-1.3129405706066049</td>
<td>2.6257152439543723</td>
</tr>
<tr>
<td>-0.2756457380259969</td>
<td>0.5511604032828874</td>
</tr>
<tr>
<td>1.6760995263084327</td>
<td>-3.3531274585380998</td>
</tr>
<tr>
<td>0.5397381405542406</td>
<td>-1.0801056482994042</td>
</tr>
<tr>
<td>-0.6565058263416250</td>
<td>1.3116177669858742</td>
</tr>
<tr>
<td>0.4333542628487280</td>
<td>-0.8690252555238786</td>
</tr>
<tr>
<td>-0.9371067286552137</td>
<td>1.8669131840428292</td>
</tr>
<tr>
<td>0.0649972739983632</td>
<td>-0.1378272311087715</td>
</tr>
<tr>
<td>-7.6398073902211787</td>
<td>14.4324223031306718</td>
</tr>
<tr>
<td>-3.7546331725690962</td>
<td>7.0777275222412030</td>
</tr>
<tr>
<td>-1.744534120970964</td>
<td>3.2564921319145139</td>
</tr>
<tr>
<td>-0.5867935086505946</td>
<td>1.0146968018735382</td>
</tr>
</tbody>
</table>

Convergence to \(-2 - i\) is now evident.
2. Once the size of the smallest circle has been set, then the next-to-smallest circle consists of those $z$-values for which $f(z)$ is on the smallest circle. In this way, each circle determines the next larger circle. Making the smallest circle smaller will make every circle smaller. The observed effect is to shift the shaded bands toward the smaller bull’s-eye.

3. For $z$ to be equidistant from $m$ and $-m$ means that $|z - m| = |z + m|$. For $f(z)$ to be equidistant from $m$ and $-m$ means that $|f(z) - m| = |f(z) + m|$. Now calculate

$$f(z) - m = \frac{1}{2} \left( z + \frac{m^2}{z} \right) - m = \frac{z^2 + m^2 - 2mz}{2z} = \frac{(z - m)^2}{2z}$$

and

$$f(z) + m = \frac{1}{2} \left( z + \frac{m^2}{z} \right) + m = \frac{z^2 + m^2 + 2mz}{2z} = \frac{(z + m)^2}{2z},$$

which implies that

$$|f(z) - m| = \frac{|z - m|^2}{2|z|} = \frac{|z + m|^2}{2|z|} = |f(z) + m|.$$ 

In other words, if $z$ is equidistant from $m$ and $-m$, then so is $f(z)$.

4. The solutions are $z = (2 + i)\text{cis}(60)$ and $z = (2 + i)\text{cis}(-60)$, as shown below:

$$\frac{1}{2} \left( z + \frac{3 + 4i}{z} \right) = \frac{1}{2}(2 + i)$$

$$z + \frac{3 + 4i}{z} = 2 + i$$

$$z^2 + 3 + 4i = (2 + i)z$$

$$z^2 - (2 + i)z + 3 + 4i = 0$$

$$z = \frac{2 + i \pm \sqrt{(2 + i)^2 - 4(3 + 4i)}}{2}$$

$$z = \frac{2 + i \pm \sqrt{(2 + i)^2 - 4(2 + i)^2}}{2}$$

$$z = \frac{2 + i \pm (2 + i)\sqrt{-3}}{2}$$

$$z = \frac{(2 + i)(1 \pm i\sqrt{3})}{2}$$

$$z = (2 + i)\text{cis}(\pm60)$$
Dynamic Fractal Solutions

page 8

1. Convert $PA = 2 \cdot PB$ into $\sqrt{(x - 3)^2 + (y - 0)^2} = 2\sqrt{(x + 3)^2 + (y - 0)^2}$, which is equivalent to $16 = (x + 5)^2 + (y - 0)^2$. The center is $(-5, 0)$ and the radius is 4.

2. The center is $(5, 0)$ and the radius is 4.

3. Convert $PA = k \cdot PB$ into $\sqrt{(x - 3)^2 + (y - 0)^2} = k\sqrt{(x + 3)^2 + (y - 0)^2}$, which is equivalent to

$$x^2 - 6x + 9 + y^2 = k^2x^2 - 6k^2x + 9k^2 + k^2y^2$$

$$0 = (k^2 - 1)x^2 + 6(k^2 + 1)x + 9(k^2 - 1) + (k^2 - 1)y^2$$

$$-9 = x^2 + 6\left(\frac{k^2 + 1}{k^2 - 1}\right)x + y^2$$

$$9 \cdot \left(\frac{k^2 + 1}{k^2 - 1}\right)^2 - 9 = \left(x + 3 \cdot \frac{k^2 + 1}{k^2 - 1}\right)^2 + y^2$$

$$9 \cdot \frac{4k^2}{(k^2 - 1)^2} = \left(x + 3 \cdot \frac{k^2 + 1}{k^2 - 1}\right)^2 + y^2$$

Thus the center is $\left(3 \cdot \frac{k^2 + 1}{1 - k^2}, 0\right)$ and the radius is $3 \cdot \left|\frac{2k}{1 - k^2}\right|$. Notice that the center approaches $(-3, 0)$ as $k$ approaches $\infty$, or $(3, 0)$ as $k$ approaches 0, and that the radius approaches 0 in either case. What happens when $k$ approaches 1?

4. Using $m = 3 + 0i$, the preceding shows that the circle center is $\frac{k^2 + 1}{1 - k^2} \cdot m$ and the radius is $\left|\frac{2k}{1 - k^2} \cdot m\right|$. This description is valid for any nonzero complex number $m$.

5. Calculate $f(z) \pm m = \frac{1}{2} \left(z + \frac{m^2}{z}\right) \pm m = \frac{z^2 \pm 2mz + m^2}{2z} = \frac{(z \pm m)^2}{2z}$. It follows that $|f(z) - m| = \frac{|z - m|^2}{2|z|}$ and $|f(z) + m| = \frac{|z + m|^2}{2|z|}$.

6. If $\frac{|z - m|}{|z + m|} = k$, then the preceding shows that $\frac{|f(z) - m|}{|f(z) + m|} = k^2$, so $j = k^2$. The root-finding process moves points on circle number $k$ inward to circle number $k^2$.

7. Given a seed value $z_0$, the resulting sequence $\{z_n\}$ therefore corresponds to a sequence $\{k_n\}$ which has the simple recursive description $k_n = k_{n-1}^2$. It follows that the value $k_0$ determines the fate of $\{z_n\}$: If $1 < k_0$, then $k_n \to \infty$ and $z_n \to -m$; if $0 < k_0 < 1$, then $k_n \to 0$ and $z_n \to m$; if $k_0 = 1$, then $z_n$ stays on the line $|z - m| = |z + m|$ forever. Even when $z_n \to m$, it need not be true that $|z_n - m| \to 0$ monotonically, however. Why?
Dynamic Fractal Solutions

page 9

1. The solutions are \( z = i \) and \( z = -1 + \frac{1}{2}i \), as shown by

\[
z = \frac{-2 + 3i \pm \sqrt{(2 - 3i)^2 + 8(1 + 2i)}}{4} = \frac{-2 + 3i \pm \sqrt{3 + 4i}}{4} = \frac{-2 + 3i \pm (2 + i)}{4}.
\]

2. The polar angle (the argument) for \( 2 + i \) is about 26.6 degrees. The argument for \( 1 + i \) is 45 degrees, hence the argument for \((2 + i)(1 + i)\) is about 26.6 + 45 = 71.6 degrees. The difference between these angles is 45 degrees.

3. The first two cases are treated by noticing that \( f(z_1) = f(z_2) \) whenever \( z_1 \cdot z_2 = 3 + 4i \). The equation \( \frac{1}{2} \left( z + \frac{3 + 4i}{z} \right) = w \) can be rewritten in quadratic form \( z^2 - 2wz + 3 + 4i = 0 \), then solved (by completing the square) to give \( z = w \pm \sqrt{w^2 - 3 - 4i} \). This shows that there are two solutions for each \( w \), unless \( w^2 = 3 + 4i \), in which case there is only one. In part (c), the solutions are \( z = 1 \) and \( z = 3 + 4i \).

4. As in item 5 on page 210, it is necessary to write

\[
f(z) = \frac{1}{3} \left( 2x + 2yi + \frac{8}{x^2 - y^2 + 2xyi} \right) = \frac{1}{3} \left( 2x + 2yi + \frac{8(x^2 - y^2 - 2xyi)}{(x^2 - y^2)^2 + (2xy)^2} \right) = \frac{1}{3} \left( 2x + 2yi + \frac{8x^2 - 8y^2 - 16xyi}{(x^2 + y^2)^2} \right),
\]

in order to isolate the functions

\[
\frac{1}{3} \left( 2x + \frac{8x^2 - 8y^2}{(x^2 + y^2)^2} \right) \text{ and } \frac{1}{3} \left( 2y - \frac{16xy}{(x^2 + y^2)^2} \right)
\]

for your calculator. These will be needed when developing a color map for the Newton-Raphson cube-root-finder, as you did for the square-root-finder on page 7.

5. You have seen many examples of attractive fixed points. For example, in item 2 on page 4, the fixed point \( \sqrt{3} \) was attractive because \( f'(\sqrt{3}) = \sqrt{3} - 2 \approx -0.268 \). On the other hand, the fixed point \(-\sqrt{3}\) for the same system was repelling because \( f'(-\sqrt{3}) = -2 - \sqrt{3} \approx -3.732 \). In item 4 on page 205, it was found experimentally that the fixed point 0.739085... for \( f(x) = \cos x \) is attractive; notice that \( \cos'(0.739085...) = -0.673612... \), by the way. All your Newton-Raphson root-finding examples have converged to superattracting fixed points. Finally, the functions \( f(x) = -x^3 - x, g(x) = -x^3 + x, \) and \( h(x) = -x \) all have \( x = 0 \) as an indifferent fixed point — given \( x_0 = 0.01 \), the \( f \)-sequence drifts very slowly away from 0, the \( g \)-sequence converges very slowly to 0, and the \( h \)-sequence settles immediately into a 2-cycle that goes nowhere.
Dynamic Fractal Solutions

6. Apply the formula of item 3 above to the case \( w = 1 - 2i \) to find that

\[
z = 1 - 2i \pm \sqrt{(1 - 2i)^2 - 3 - 4i} = 1 - 2i \pm \sqrt{-6 - 8i} = 1 - 2i \pm (2 + i)\sqrt{2} = 1 \pm \sqrt{2} + (-2 \pm 2\sqrt{2})i,
\]

which are approximately \(-0.4142136 + 0.8284272i\) and \(2.4142136 - 4.8284272i\). Notice that both answers satisfy \( y = -2x \). Because \( f(1 - 2i) = 0 \) and \( f(0) = \infty \), these \( z \)-values initiate sequences that lead eventually to \( \infty \) (as would the two solutions to the equation \( f(z) = -1 + 2i \), by the way).

7. This is an imaginary application of the slow square-root finder introduced in item 2 on page 3. You should expect that the dynamic system defined by

\[
f(z) = \frac{z + m^2}{z + 1} \quad \text{and} \quad f'(z) = \frac{1 - m^2}{(z + 1)^2}
\]

will have fixed points at \( m \) and \(-m\), one attracting and the other repelling. To check your intuition, examine the illustrated case \( m = 2 + i \): Because

\[
|f'(2 + i)| = \left| \frac{-2 - i}{5} \right| = \frac{1}{5} \sqrt{5} = 0.44721 \ldots,
\]

it follows that \( 2 + i \) is an attracting fixed point of \( f \). Because

\[
|f'(-2 - i)| = |-2 + i| = \sqrt{5} = 2.23606 \ldots,
\]

it follows that \(-2 - i\) is a repelling fixed point of \( f \). This is why there is only one system of shaded bands in the figure — with the exception of the sequence that is fixed at \(-2 - i\), all sequences move away from \(-2 - i\) and toward \( 2 + i \).

8. The \( z \)-values in question are those that lead to \( \infty \). First comes \(-1\), because \( f(-1) = \infty \). The next one is \(-2 - 2i\), because \( f(-2 - 2i) = -1 \). The next one is \( \frac{1}{13}(27 + 8i)\), because \( f(\frac{1}{13}(27 + 8i)) = -2 - 2i \). It so happens that these \( z \)-values are not as troublesome as you might expect. The reason is that \( f(\infty) = 1 \). Thus, unless your calculator has been programmed to shut down when presented with division by zero, the sequence continues with no apparent interruption. This is why the figure shows no trace of disturbance near \(-1\) or \(-2 - 2i\), or any of the other precursors of \( \infty \). Incidentally, did you notice that this sequence of \( z \)-values approaches \(-2 - i\) as a limit?

9. Some examples of invariant sets: the half-plane above the line; the half-plane below the line; either of the single fixed points; any \( \text{orbit} \ \{z_0, z_1, z_2, \ldots\} \); the set of all numbers \( r(2 + i) \) for which \( r \) is a positive real number.

May 2006
1. Newton-Raphson square-root finder:
   (a) Because the seed point has rational (integer) coordinates, every term $z_n$ of the sequence must have rational coordinates. Thus the sequence misses all the points on the line $y = -2x$ that have irrational coordinates — for example, $(\sqrt{2}, -\sqrt{8})$, or the points found in item 6 on page 219. It so happens that most of the points on the line $y = -2x$ are irrational points. Nevertheless, it is possible to show that the given sequence is an everywhere dense subset of the line, meaning that it visits every segment of positive length, no matter how small!
   (b) Points that are not on the line initiate sequences that converge to one of the two roots (see item 1 on page 11), whereas the ancestral points under discussion initiate sequences that, by definition, lead to 0.

2. A web diagram is shown at right. It is evident that $x_1$ is between $-1$ and 1, inclusive, no matter what $x_0$ is. If $0 < x_1$, then the sequence decreases because $\sin x < x$. A decreasing sequence of positive numbers must converge to something, and the limit can only be 0 because the sine function is continuous and $x = 0$ is its only fixed point. Because $|x_{n+1} - 0| = |\sin x_n| \leq |x_n - 0|$, the convergence is linear, and it is excruciatingly slow. For example, suppose $x_0 = 1$. It is found that
   
   $x_{295} = 0.099858023198$ is the first term less than 0.1,
   $x_{29992} = 0.009999897363$ is the first term less than 0.01,
   $x_{2999989} = 0.000999999937$ is the first term less than 0.001,

   suggesting that it takes about $3 \times 10^{2k}$ terms for the sequence to drop below $10^{-k}$.

3. The fixed point is $x = \frac{1}{2}$, which marks a tangency between the parabola and the line $y = x$. The increasing sequence seeded by $x_0 = 0$ converges to 0.5 very slowly — it takes 94 terms to pass 0.49, 992 terms to pass 0.499, and 9990 terms to pass 0.4999, suggesting that each decimal place takes about ten times as long to establish as the preceding one.

   On the other hand, the sequence seeded by $x_0 = 0.5000000001$ is also increasing, and therefore it diverges to infinity. It takes a long time to pry itself away from $x = 0.5$, however — probably about $10^{10}$ terms are needed for 0.6 to be passed. Most calculating devices are incapable of giving the correct value for $x_1$, by the way. To a TI-83, this is a constant sequence!

4. The solutions to $|z - m| < |z + m|$ are those $z$ that are closer to $m$ than to $-m$; they form a half-plane. The other inequality defines the half-plane of $z$-values that are closer to $-m$ than to $m$. 

May 2006

220

Phillips Exeter Academy
5. Figure 3 shows the color map for the dynamic system defined by the Newton-Raphson cube-root-finding function \( f(z) = \frac{1}{3}(2z + 8z^{-2}) \). Points that are coded white are all attracted to the real root 2; they form the \textit{basin of attraction} of this root. Points that are coded black form the basin of attraction of the root \(-1 + i\sqrt{3}\). Points that are coded gray form the basin of attraction of the root \(-1 - i\sqrt{3}\). The central point in the figure is 0 (the origin), and the width of the frame is 6.

This startling picture is an unexpected development in our investigation of the root-finding process. What makes this example even more interesting is its history, for this represents the first well-studied problem whose solution exhibits what is now called a \textit{fractal}. In 1879, the British mathematician Arthur Cayley proposed to study the dynamic behavior of complex numbers when the Newton-Raphson cube-root method is applied to them. In effect, his goal was to understand and describe the patterns you can now see in Figure 3.

Having already disposed easily of square roots, Cayley may have felt confident that he would be successful with this problem; however, he never did publish any results. It was not until forty years later that two French mathematicians, Pierre Fatou and Gaston Julia, independently solved Cayley’s problem. Neither man lived long enough to see computer-generated images of his work, although Julia lived until 1978, the year before the advent of personal computers.

The indecisive seed values whose sequences are attracted to no root form the \textit{Julia set} of the dynamic system. Among other things, Julia proved that \textit{any such point must lie on the border of all three basins of attraction}.

Figure 4 displays the dynamics of the root-finding process by means of bands of alternating color. In the interest of clarity, only the attracting basin of 2 is shown. Given any sequence \( \{z_n\} \) that approaches 2, successive terms lie in bands of opposite colors. The tiny bull’s-eye is a circle of radius 0.01 centered at 2.
Dynamic Fractal Solutions

page 11

1. The root $2 + i$ is not between

$$z_0 = 1 \quad \text{and} \quad \frac{3+4i}{z_0} = 3 + 4i,$$

because these three points are not collinear. Nor is $2 + i$ between

$$z_1 = 2 + 2i \quad \text{and} \quad \frac{3+4i}{z_1} = \frac{7}{4} + \frac{1}{4}i.$$

In general, $z_n$ and $(3+4i)z_n^{-1}$ are not collinear with $2 + i$. It is true, however, that $|2 + i|$ is always between $|z_n|$ and $|(3 + 4i)z_n^{-1}|$ — why?

2. To show that $|f(z) - m| < \frac{1}{2}|m|$ whenever $|z - m| < \frac{1}{2}|m|$, first notice that every point inside the circle $|z - m| = \frac{1}{2}|m|$ is further from the origin than $\frac{1}{2}m$ is. It follows that

$$|f(z) - m| = \frac{|z - m|^2}{2|z|} = |z - m| \frac{|z - m|}{2|z|} < |z - m| \frac{\frac{1}{2}|m|}{\frac{1}{2}|m|} = \frac{1}{2}|z - m|.$$

The image $f(z)$ is closer to the center $m$ than $z$ is, so the circle interior is invariant.

3. You are given $|z_0 - m| < \frac{1}{3}|m|$. Because the entire orbit $z_0 , z_1 , z_2 , \ldots$ is enclosed by the circle (invariance), the preceding discussion demonstrates that $|z_n - m| < \frac{1}{3}|z_{n-1} - m|$ holds for all positive $n$.

4. To show that $|f(z) - m| < \frac{2}{3}|m|$ whenever $|z - m| < \frac{2}{3}|m|$, first notice that every point inside the circle $|z - m| = \frac{2}{3}|m|$ is further from the origin than $\frac{1}{2}m$ is. It follows that

$$|f(z) - m| = \frac{|z - m|^2}{2|z|} = |z - m| \frac{|z - m|}{2|z|} < |z - m| \frac{\frac{2}{3}|m|}{\frac{2}{3}|m|} = |z - m|.$$

The image $f(z)$ is closer to the center $m$ than $z$ is, so the circle interior is invariant.

5. Suppose that $|z_0 - m| \leq k|m|$, where $0 < k < \frac{2}{3}$. Reasoning as in the preceding item, observe that

$$|z_1 - m| = \frac{|z_0 - m|^2}{2|z|} = |z_0 - m| \frac{|z_0 - m|}{2|z_0|} < |z_0 - m| \frac{k|m|}{2(1-k)|m|} = \frac{k}{2 - 2k}|z_0 - m|.$$

Thus the disk $|z - m| \leq k|m|$ is invariant. For all positive $n$, inductively conclude that $|z_n - m| < \left( \frac{k}{2 - 2k} \right)^n |z_0 - m|$, which approaches 0 because $k < \frac{2}{3}$.

6. Because $z_1 = \frac{1}{3}(-1+2i)\sqrt{3}$ and $z_2 = z_0$, the orbit is a 2-cycle. You have seen a 2-cycle before — in item 6 on page 1. Item 3 on page 12 presents another.
Dynamic Fractal Solutions

page 12

1. To find the ancestors of zero, solve $N(z) = 0$, which can be rewritten in the form $z^3 = -4$, then solved routinely using polar form. On the other hand, finding the ancestors of a nonzero complex number $w$ by solving $N(z) = w$ is not usually a simple process, for you are confronted by the cubic equation $2z^3 - 3wz^2 + 8 = 0$. Unless $w$ is special in some way (see item 4, for example), it is unlikely that factoring will succeed. Instead, numerical approximations usually suffice. For those who delight in mathematical curiosities, there is also the cubic formula (see page 13), which delivers the following three immediate ancestors of $-\sqrt[3]{4}$:

$$-\frac{1}{2} \sqrt[3]{4} - \frac{3}{2} - \sqrt{6} - \frac{3}{2} + \sqrt{6}$$

$$\frac{1}{2} \left( -\sqrt[3]{4} + \frac{3}{2} - \sqrt{6} + \frac{3}{2} + \sqrt{6} \right) + \frac{\sqrt{3}}{2} \left( -\frac{3}{2} - \sqrt{6} + \frac{3}{2} + \sqrt{6} \right) i$$

$$\frac{1}{2} \left( -\sqrt[3]{4} + \frac{3}{2} - \sqrt{6} + \frac{3}{2} + \sqrt{6} \right) - \frac{\sqrt{3}}{2} \left( -\frac{3}{2} - \sqrt{6} + \frac{3}{2} + \sqrt{6} \right) i$$

Approximately, these are $z = -2.867550999$ and $z = 0.243224710 \pm 1.155750952i$. Can you find them in Figure 3?

2. The preceding discussion shows that every complex number $w$ has three ancestors, for the equation $N(z) = w$ is equivalent to the cubic equation $2z^3 - 3wz^2 + 8 = 0$. Cubic equations always have three solutions, although it can happen that one of the solutions is repeated (see item 4 below). That a polynomial equation of degree $n$ always has $n$ solutions among the complex numbers is called the Fundamental Theorem of Algebra — a reassuring result that is simple to state but difficult to prove.

3. These two $z$-values were found by solving $N(N(z)) = z$, which can be simplified to the ninth-degree equation $5z^9 - 30z^6 - 48z^3 - 256 = 0$. This is not as bad as it looks, for included among the nine solutions are the three fixed points. In other words, you can factor the equation as $(z^3 - 8)(5z^6 + 10z^3 + 32) = 0$. The 2-cycle comes from $5z^6 + 10z^3 + 32 = 0$, which is of quadratic type. What is the significance of the other four solutions?

4. Write the cube-root finder in the form $N(z) = \frac{1}{3}(2z + m^3z^{-2})$. The ancestors of any root $m$ are found by solving $N(z) = m$, which (as above) is equivalent to the cubic equation $2z^3 - 3mz^2 + m^3 = 0$. Knowing that $m$ is a fixed point of $N$ tells you that $z - m$ must be a factor of this equation — indeed, it must be a double factor, because $m$ is a superattractor of $N$. The equation factors $(z - m)^2(2z + m) = 0$, so $z = -\frac{1}{2}m$ is the ancestor you seek. The complete ancestral tree of $m = 2$ shows prominently in Figure 4 — do you see it?

5. If $z_0$ equals an eventually fixed point, then the sequence $\{z_n\}$ becomes constant — i.e., there are only finitely many different terms in it. You have not seen this phenomenon before, because it occurs only when the Julia set divides a (monochromatic) basin of attraction into components.
Dynamic Fractal Solutions

page 13

7. Cubic formula calculations:
(a) Because the coefficient of $z^2$ in $z^3 - 6z + 4$ is zero, it is straightforward to calculate $d = -4$, hence $p = \sqrt[3]{-2 + 2i}$ and $q = -\sqrt[3]{2 + 2i}$. These simplify to $p = 1 + i$ and $q = 1 - i$.

The three roots are therefore $z = 2$, $z = -1 + \sqrt{3}$, and $z = -1 - \sqrt{3}$. The given equation could have been put into factored form $(z - 2)(z^2 + 2z - 2) = 0$.

(b) Substitute $w = z + 1$, which leads to

$0 = w^3 - 3w^2 - 9w + 23$

$0 = (z + 1)^3 - 3(z + 1)^2 - 9(z + 1) + 23$

$0 = (z^3 + 3z^2 + 3z + 1) - 3(z^2 + 2z + 1) - 9(z + 1) + 23$

$0 = z^3 - 12z + 12$

Now calculate $d = -28$, $p = \sqrt[3]{-6 + i\sqrt{28}}$, and $q = \sqrt[3]{-6 - i\sqrt{28}}$, with $p$ in the first quadrant and $q$ in the fourth. The three roots are

$z_1 = \sqrt[3]{-6 + i\sqrt{28}} + \sqrt[3]{-6 - i\sqrt{28}}$

$z_2 = -\frac{1}{2}z_1 + \frac{1}{2} \left( \sqrt[3]{-6 + i\sqrt{28}} - \sqrt[3]{-6 - i\sqrt{28}} \right)i\sqrt{3}$

$z_3 = -\frac{1}{2}z_1 - \frac{1}{2} \left( \sqrt[3]{-6 + i\sqrt{28}} - \sqrt[3]{-6 - i\sqrt{28}} \right)i\sqrt{3}$

The corresponding $w$-values are obtained by adding 1 to each of the preceding. They are all real. The first one is $w_1 = 1 + 2\cos \left( \frac{1}{3} \text{ arctan} \frac{1}{2} \sqrt{7} \right) - 2\sqrt{3} \sin \left( \frac{1}{3} \text{ arctan} \frac{1}{2} \sqrt{7} \right) \approx 2.115749 \ldots$

(c) Notice that the equation factors $0 = (w + 1)(w^2 + 1)$, so that the three roots are $-1$, $i$, and $-i$. Or, you can substitute $w = z - \frac{1}{3}$, which leads to

$0 = w^3 + w^2 + w + 1$

$0 = (z - \frac{1}{3})^3 + (z - \frac{1}{3})^2 + (z - \frac{1}{3}) + 1$

$0 = z^3 + \frac{2}{3}z + \frac{20}{27}$

so that $d = \frac{4}{27}$, $p = \frac{1}{3}\sqrt[3]{-10 + \sqrt{108}}$, $q = \frac{1}{3}\sqrt[3]{-10 - \sqrt{108}}$, $z_1 = -\frac{2}{3}$, and so forth.
1. The Newton-Raphson function associated with an equation \( E(z) = 0 \) is

\[
N(z) = z - \frac{E(z)}{E'(z)} = \frac{zE'(z) - E(z)}{E'(z)}.
\]

If \( E \) is a polynomial, then \( N \) is a rational function in which the degree of the numerator is one greater than the degree of the denominator. Thus \( N(\infty) = \infty \) makes sense. To say that \( \infty \) is a repelling fixed point means simply that most \( z \)-values that are very far from the origin still initiate sequences that converge to (finite) fixed points for \( N \) (i.e., to solutions of \( E(z) = 0 \)). In contrast, the slow root-finder \( f \) examined in item 7 on page 9 does not have infinity as a fixed point — instead, \( f(\infty) = 1 \). In this example, then, \( \infty \) can occur as just another term in a sequence that is converging to \( m \).

2. Given any term \( z_n \) of the orbit, \( f(z_n) \) is simply the next term of the orbit.

3. Yes — any number of terms is possible. An orbit is finite if it is (eventually) periodic. This means that \( z_n = z_m \) occurs for some integers \( m \) and \( n \). If \( z_n = z_0 \), then the orbit is purely periodic, as in the 2-cycle example in item 6 on page 1 (set \( x_0 = 1/\sqrt{3} \)). If \( z_0 \) is not part of the cycle, then the orbit is eventually periodic — for example, let \( z_0 \) be any ancestor of a fixed point.

4. As the figure at right shows, the graph of the function \( N(x) = \frac{1}{3}(2x + 8x^{-2}) \) is asymptotic to the linear graph \( y = \frac{2}{3}x \). This implies that each large (negative) real ancestor of 0 will be approximately \( \frac{3}{2} \) times its immediate successor to its right. It follows that the spaces between these \( x \)-values will also increase by approximately a factor of \( \frac{3}{2} \).

5. The Figure 5 shows the Julia set for the Newton-Raphson fourth-root finder

\[
N(z) = \frac{1}{4} \left( 3z + \frac{m^4}{z^3} \right).
\]

The value of \( m^4 \) is real and positive. The only other thing that can be inferred about \( m \) is its approximate location in the figure. The simplest approach is to find the immediate ancestors of 0, by solving \( N(z) = 0 \), or \( 3z^4 = m^4 \). One of the four ancestors is

\[
\frac{m}{\sqrt{3}} \cis(45) = (0.537 + 0.537i)m.
\]

Another approach (used in item 4 on page 223) is to find the immediate ancestors of \( m \), by solving \( \hat{N}(z) = m \), or \( 0 = 3z^4 - 4mz^3 + m^4 = (z - m)^2(3z^2 + 2mz + m^2) \). This leads to the solutions \( z = m \) and \( z = \frac{1}{3}m(-1 \pm i\sqrt{2}) \); Notice that there are two ancestors for \( m \) other than \( m \) itself. The width of the figure is approximately \( 3|m| \).
6. Item 3 on page 223 shows that the Julia set contains a 2-cycle (three of them, in fact). Such cycles can not be ancestral to 0 — or to any other point, for that matter. As you will see, the Julia set contains many cycles, as well as infinitely many other hard-to-describe and hard-to-find points.

7. Start by evaluating the left-hand side:

\[ N(z \text{cis}(120)) = \frac{1}{3} \left( 2z \text{cis}(120) + \frac{8}{|z \text{cis}(120)|^2} \right) \]
\[ = \frac{1}{3} \left( 2z \text{cis}(120) + \frac{8}{z^2 \text{cis}(240)} \right) \]
\[ = \frac{1}{3} \left( 2z \text{cis}(120) + \frac{8 \text{cis}(120)}{z^2} \right) \]
\[ = \frac{1}{3} \left( 2z + \frac{8}{z} \right) \text{cis}(120) \]
\[ = N(z) \text{cis}(120), \]

and the right-hand side appears. This identity shows that points that differ only by a 120-degree rotation will exhibit similar behavior — their orbits will be congruent. This explains why the figure displays threefold rotational symmetry. In effect, the cube-root system is made up of three equivalent invariant subsystems, which are called isomorphic.

8. A small portion of the figure centered at \( z \) is transformed by \( N \) into a small portion of the figure centered at \( N(z) \). Except for some mild distortion and rotation — which is prescribed by the magnitude and argument of \( N'(z) \) — both portions should look the same.

9. To decide whether the root \( m \) is an attractor, calculate (see item 7 on page 219)

\[ |f'(m)| = \left| \frac{1 - m^2}{(m + 1)^2} \right| = \frac{|1 - m|}{|1 + m|}. \]

Whether \( m \) is an attractor depends simply on whether \( |f'(m)| < 1 \). This is equivalent to asking about the inequality \( |1 - m| < |1 + m| \), whose solutions consist of \( m \)-values that are closer to 1 than to \(-1\). In other words, the dynamic system has \( m \) as an attractor exactly when \( m \) has positive real part.

To make \(-m\) the attractor, use \( g(z) = \frac{m^2 - z}{z - 1} \) as the system function. Notice that this is just \(-f(-z)\). Was this predictable?
1. The entries in the $e_n$ column show that the distance from $z_n$ to $m$ does not decrease monotonically. The convergence of this sequence is therefore very much in doubt.

It is true that $z_0$ is closer to $2 + i$ than it is to zero, because $58.6 < |z_0|$. The proposition is therefore false.

The calculations are correct, however. In particular, it is true that

$$|z_{n+1} - m| < \frac{|z_n - m|}{2}$$

whenever $|z_n - m| < |z_n|$. It need not be true, however, that $|z_{n+1} - m| < |z_{n+1}|$. In other words, even though $z_n$ is more than twice as far from $m$ as $z_{n+1}$ is, it is not because $z_{n+1}$ is between $z_n$ and $m$. As the example shows, it is still possible for $z_{n+1}$ to be closer to zero than to $m$. This of course invalidates the next inequality.

Thus the statement “one can inductively assume that $|z_n - m| < |z_n|$” is the faulty step.

Although the proposition is false, it is nevertheless true that all the sequences in question do approach $m$. Because the convergence is not monotonic, the proof of convergence has to be based on different reasoning.
1. Solve $z^2 = z$ to find the two finite fixed points $z = 0$ and $z = 1$. Because $Q'(0) = 0$, the origin is a superattracting fixed point. Because $Q'(1) = 2$, the fixed point at 1 is repelling.

To say that $\infty$ is an attracting fixed point means the following: If $|z|$ is large, then $|z^2|$ is larger. (On the complex sphere, squaring moves points in the upper hemisphere closer to the north pole; see next page.)

2. If $|z_0| < 1$, then $z_n \to 0$. If $1 < |z_0|$, then $z_n \to \infty$. If $|z_0| = 1$, then $z_n$ approaches neither attractor.

3. Solve $z_2 = z_0$, which is $z_0^3 = z_0$, or $z_0(z_0^3 - 1) = 0$. The two nonreal cube roots of 1 are the numbers you want: $z_0$ can be either cis(120) or cis(240).

4. Solve $z_3 = z_0$, which is $z_0^8 = z_0$, or $z_0(z_0^7 - 1) = 0$. There are six nonreal $7^{th}$ roots of 1, and they group into two 3-cycles: One cycle is

$$\{ \text{cis}\left(\frac{360}{7}\right), \text{cis}\left(\frac{720}{7}\right), \text{cis}\left(\frac{1440}{7}\right)\}$$

and the other cycle is

$$\{ \text{cis}\left(\frac{1080}{7}\right), \text{cis}\left(\frac{2160}{7}\right), \text{cis}\left(\frac{1800}{7}\right)\}.$$ 

5. If $\{z_n\}$ is a nonconstant $k$-cycle, then it can not approach either of the attractors.

6. $Q$ has three 4-cycles, which are found by solving $Q^4(z) = z$, which is $z^{16} = z$, or $z(z^{15} - 1) = 0$. The nonreal $15^{th}$ roots of 1 can be arranged as three 4-cycles and one 2-cycle:

$$\{\text{cis}(24), \text{cis}(48), \text{cis}(96), \text{cis}(192)\}$$
$$\{\text{cis}(72), \text{cis}(144), \text{cis}(288), \text{cis}(216)\}$$
$$\{\text{cis}(168), \text{cis}(336), \text{cis}(312), \text{cis}(264)\}$$
$$\{\text{cis}(120), \text{cis}(240)\}$$

7. To find $k$-cycles, solve the equation $Q^k(z) = z$, which is $z^{2^k} = z$, or $z(z^{2^k-1} - 1) = 0$. One of the $k$-cycles is generated by $z_0 = \text{cis}\left(\frac{360}{2^k-1}\right)$. Notice, by the way, that the $k$-cycle equation includes among its $2^k$ solutions all $j$-cycle elements for which $j$ divides $k$.

8. In this dynamic system, finding the ancestors of a complex number means finding its two square roots, the square roots of those square roots, and so on. The ancestors of 1 are: 1 and $-1$; $i$ and $-i$; cis(45), cis(135), cis(225), and cis(315); ... in general, there are $2^{k-1}$ ancestors that occur $k$ generations before 1 — namely, the odd powers of cis$\left(\frac{360}{2^k}\right)$.
9. Any nonzero complex number $z$ has two square roots. If $z$ is part of a 4-cycle, then one of the square roots precedes $z$ in the cycle. The other square root leads to the cycle without being part of it. The square roots of this root have the same property, and so forth — they are all eventually periodic. For example, $\text{cis}(24)$ is part of a 2-cycle, and one of its square roots is $\text{cis}(192)$, which is the preceding term in the cycle. The other square root of $\text{cis}(24)$ is $\text{cis}(12)$, which is therefore eventually 4-periodic. Its square roots, $\text{cis}(6)$ and $\text{cis}(186)$, have the same property, as do their square roots, etc.

10. You have enumerated infinitely many points on the unit circle, but they have all been of the form $\text{cis}(360t)$, where $t$ is a rational number. Just let $t = \frac{1}{2}\sqrt{2}$ to obtain a dramatically different example.

11. This is answered in item 1 on page 228. Sequences that approach the origin do so quadratically — what else? (See item 1 on page 201).

12. Let $z_0 = \text{cis}(73.654)$. My calculator shows the following values for $z_{60}$, $z_{61}$, and $z_{62}$:

<table>
<thead>
<tr>
<th>$x_n$</th>
<th>$y_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.02563905</td>
<td>1.05750855</td>
</tr>
<tr>
<td>-1.11766698</td>
<td>0.05422703</td>
</tr>
<tr>
<td>1.24623890</td>
<td>-0.12121553</td>
</tr>
</tbody>
</table>

Notice that the sequence has fallen outside the Julia set. You have seen this phenomenon before (item 1 on page 215, for example).

13. Finding ancestors means finding repeated square roots. If $z$ is outside the unit circle, these repeated roots will lie closer and closer to the unit circle. The same is true for any nonzero $z$ inside the unit circle. Except for the attractors, therefore, ancestral trees lead you toward the Julia set.

14. Because the derivative of the linear function is $m$, there can be convergence only when $|m| < 1$. In this case, the attractor is the fixed point $z = \frac{b}{1 - m}$. Rewrite the linear function

$$mz + b = mz + (1 - m) \frac{b}{1 - m} = \frac{b}{1 - m} + m \left( z - \frac{b}{1 - m} \right)$$

to see how successive errors are diminished at the constant rate $|m|$. This is linear convergence, of course. Is it a coincidence that the formula for the fixed point looks like the sum of an infinite geometric series?
Dynamic Fractal Solutions

1. Given $a$ and $b$,

$$p = \frac{2a}{a^2 + b^2 + 1}, \quad q = \frac{2b}{a^2 + b^2 + 1}, \quad \text{and} \quad r = \frac{a^2 + b^2 - 1}{a^2 + b^2 + 1},$$

Inversely, given $p$, $q$, and $r$,

$$a = \frac{p}{1 - r} \quad \text{and} \quad b = \frac{q}{1 - r}.$$

2. The inverses 2 and $\frac{1}{2}$ correspond to $(\frac{1}{5}, 0, \frac{3}{5})$ and $(\frac{1}{5}, 0, -\frac{3}{5})$, respectively; the inverses $i$ and $-i$ correspond to $(0, 1, 0)$ and $(0, -1, 0)$, respectively; the inverses $2 + i$ and $\frac{2}{5} - \frac{1}{5}i$ correspond to $(\frac{2}{3}, 1, \frac{2}{3})$ and $(\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3})$, respectively; the inverses $-1 + i$ and $-\frac{1}{2} - \frac{1}{2}i$ correspond to $(-\frac{3}{5}, \frac{2}{5}, \frac{1}{5})$ and $(-\frac{2}{5}, -\frac{2}{5}, -\frac{1}{5})$, respectively. In all cases, inverse pairs correspond to $(p, q, r)$ and $(p, -q, -r)$.

3. Given a straight line in the complex plane, there is a unique plane that contains both the given line and the pole $(0,0,1)$. This plane intersects the sphere along a circle through the pole; thus all straight lines go through $\infty$. Given a circle $a = \alpha + \rho \cos \theta$, $b = \beta + \rho \sin \theta$ in the complex plane, the corresponding points on the complex sphere all lie in the plane

$$2\alpha p + 2\beta q + (\alpha^2 + \beta^2 - 1 - \rho^2)r = \alpha^2 + \beta^2 + 1 - \rho^2,$$

as a long calculation shows. For example, the circle $a = 8 + 5 \cos \theta$, $b = 6 + 5 \sin \theta$ corresponds to the points where the plane $8p + 6q + 37r = 38$ intersects the complex sphere $p^2 + q^2 + r^2 = 1$.

4. This is the pattern observed in item 2. The inverse of $(a, b)$ is $\left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2}\right)$, which corresponds to (after simplification)

$$\left(\frac{2a}{a^2 + b^2 + 1}, \frac{-2b}{a^2 + b^2 + 1}, \frac{1 - a^2 - b^2}{a^2 + b^2 + 1}\right)$$

on the complex sphere. Compare with item 1.

5. Multiplication by $i$ is represented by a quarter-turn of the complex sphere about the zero-infinity axis.

6. Multivariable calculus can be used to show that vectors $[1,0]$ and $[0,1]$ at $(a, b)$ map to

$$\frac{2}{(a^2 + b^2 + 1)^2}[1 + b^2 - a^2, -2ab, 2a] \quad \text{and} \quad \frac{2}{(a^2 + b^2 + 1)^2}[-2ab, 1 + a^2 - b^2, 2b]$$

on the complex sphere. These are perpendicular, and have length $2(1 + a^2 + b^2)^{-1}$. 

May 2006

230

Phillips Exeter Academy
1. If \( f(\{f(z)\}) = z \) for all \( z \), then \( f \) is its own inverse. One example is \( f(z) = 5z^{-1} \); another is \( f(z) = 2 - z \).

2. First calculate

\[
N'(w) = \frac{2}{3} - \frac{16}{3w^3} \quad \text{and} \quad 3w^3 = 4.943 \text{cis}(234.36),
\]

then use the \( \text{cis}(\theta)^{-1} = \text{cis}(-\theta) \) identity to obtain

\[
N'(w) = 0.667 - 3.237 \text{cis}(125.64) = 2.553 - 2.631i = 3.666 \text{cis}(-45.86),
\]

which shows the local magnification factor 3.666 and the local turning angle \(-45.86\).

3. Doubling modulo 1:
(a) \( a_5 = 8\pi - 25 = 0.1327\ldots \)
(b) Let \( a_0 \) be any dyadic rational — a number of the form \( \frac{m}{2^n} \), with \( m \) and \( n \) integers.
(c) One possibility is \( a_0 = \frac{1}{31} \); there are many others.

4. Any irrational choice for \( a_0 \) will do — as in item 3a. Because of their limited precision, calculators and computers can manipulate only rational numbers, so such sequences would have to repeat eventually.

5. One example is \( z_0 = \text{cis}(120) \).
(b) A dense subset of the circle intersects every arc of the circle, no matter how small. There are ancestors of 1 everywhere you look.
(c) Points of the form \( \text{cis}(2^{-n}180) \) cluster at 1, but nowhere else.
(d) If the fraction \( \frac{m}{2n+1} \) is in lowest terms — with \( m \) and \( n \) positive integers — then
\[
\text{cis}\left(\frac{m}{2n+1}360\right)
\]

is a non-fixed periodic point. The set of all such points is dense in the Julia set. Another possibility is to consider the ancestral tree of (say) a period-2 point.
(e) Such a sequence could approach \( \infty \) (if \( z_0 = 1.01 \)) or zero (if \( z_0 = 0.99 \)); it could also wander for a long time in the Julia set (if \( z_0 = \text{cis}(0.01) \)).

6. Limits of complex sequences:
(a) This sequence approaches zero along the spiral path \( r = 2^{-\theta/90} \).
(b) Once this sequence reaches the real axis (after two iterations), it stays on it. There is no spiral.
Dynamic Fractal Solutions

1. The arithmetic of doubling angles is presented in item 3 on page 19. If \( a_0 = \frac{k}{m} \), with \( k \) and \( m \) positive integers, then all subsequent values \( a_n \) will be positive rationals, with denominators that are no greater than \( m \). There are fewer than \( m \) such fractions possible, so the sequence \( \{a_n\} \) must eventually repeat itself. Periodic examples occur when the initial fraction can be written with an odd denominator, and eventually periodic examples account for the rest. Eventually fixed means that some \( a_n \) is zero, which happens only if the initial fraction can be written with a power of 2 as the denominator.

2. If two terms of this sequence had the same value, then \( 2^m \sqrt{2} - p = 2^n \sqrt{2} - q \), for some integers \( m, n, p, \) and \( q \). Rearrange the equation to read

\[
\sqrt{2} = \frac{p - q}{2^m - 2^n}.
\]

This is an impossibility, for it says that \( \sqrt{2} \) is rational.

3. As noted in the previous item, the irrationality of \( a_0 \) is what prevents periodicity from occurring.

4. The finite fixed points are found by solving \( z = z^2 - \frac{5}{16} \), or \( 16z^2 - 16z - 5 = 0 \). The solutions are \( z = \frac{5}{4} \) and \( z = -\frac{1}{4} \). Because \( Q_c'(z) = 2z \) (no matter what \( c \) is), it is easy to see that \( z = -\frac{1}{4} \) is attracting, while \( z = \frac{5}{4} \) is repelling. (b) Repelling fixed points must be in the Julia set because they do not seed orbits that approach an attractor. On the other hand, the attracting fixed point must be surrounded by the Julia set, which separates complex numbers into two two basins — one consisting of those complex numbers whose orbits approach \( \infty \), the other consisting of those complex numbers whose orbits approach the attracting finite fixed point. (c) The immediate ancestors of \( \frac{5}{4} \) are \( \pm \frac{3}{4} \), and the parents of \( -\frac{5}{4} \) are found by solving the equation \( z^2 - \frac{5}{16} = -\frac{5}{4} \); they are \( \pm \frac{1}{4} i \sqrt{15} \).

5. Solve \( z = z^2 + c \), using the quadratic formula.

6. Because the sum of the fixed points is 1 (see preceding item), the other fixed point must be \( \frac{1}{4}(5 - i) \). The first one is attracting, for \( |\frac{1}{2}(-1 + i)| = \frac{\sqrt{5}}{2} \), which is less than 1. The second fixed point is repelling, for \( |\frac{1}{2}(5 - i)| = \frac{\sqrt{26}}{2} \), which is greater than 1.

7. Given that \( \frac{1}{2} i = [\frac{1}{2} i]^2 + c \), it follows that \( \frac{1}{2} i - [\frac{1}{2} i]^2 = c \), or \( \frac{1}{2} i + \frac{1}{4} = c \). The other finite fixed point is \( 1 - \frac{1}{2} i \). The first fixed point is indifferent, for \( Q_c'\left(\frac{1}{2} i\right) = i \), whose magnitude is 1. The second fixed point is repelling, for \( Q_c'\left(1 - \frac{1}{2} i\right) = 2 - i \), whose magnitude is \( \sqrt{5} \).
1. Suppose that $2 < |z_0|$, and consider the magnitude of $z_1 = Q_c(z_0) = z_0^2 - \frac{5}{16}$. The smallest possible magnitude for $z_1$ would occur if the vectors $z_0^2$ and $-\frac{5}{16}$ were parallel, thus

$$|z_1| = |z_0^2 - \frac{5}{16}| \geq |z_0^2| - \frac{5}{16}.$$ 

It follows that

$$\frac{|z_1|}{|z_0|} \geq \frac{|z_0^2| - \frac{5}{16}}{|z_0|} = |z_0| - \frac{\frac{5}{16}}{|z_0|} > 2 - \frac{5}{32} = \frac{59}{32}.$$ 

In other words, $|z_1|$ is guaranteed to be more than $\frac{59}{32}$ times as large as $|z_0|$. Because $2 < |z_1|$ is now known to be true (i.e., the circle exterior $2 < |z|$ is invariant), the same reasoning can be applied to $z_1$. Inductively, obtain

$$|z_{n+1}| > \left(\frac{59}{32}\right)^n |z_0|,$$ 

proving that $|z_n|$ approaches $\infty$ monotonically.

2. Because the outer contour is a circle, and because the squaring process is quadratic, it is not unreasonable to wonder whether the oval (shown at right) is a *bona fide* ellipse. It is not. To see why, let $(x, y)$ be a typical point on the oval, so that $(x + yi)^2 - \frac{5}{16}$ is on the circle of radius 2. Thus

$$(x^2 - y^2 - \frac{5}{16})^2 + (2xy)^2 = 4,$$ 

whose axis intercepts are found to be

$$x = \pm \frac{1}{4}\sqrt{37} \quad \text{and} \quad y = \pm \frac{1}{4}\sqrt{27}.$$ 

Because the oval has symmetry with respect to both coordinate axes, its equation would have to be

$$\frac{16}{37}x^2 + \frac{16}{27}y^2 = 1,$$ 

if it were actually an ellipse. In particular, the points $(1, \pm \sqrt{\frac{592}{567}}) = (1, \pm 1.0218)$ would have to be on the oval. Instead, it is routine to calculate (the equation can be treated as a quadratic in $y^2$) that $y = \pm \frac{1}{4}\sqrt{-21 + \sqrt{1344}} = \pm 0.9893$ when $x = 1$.

3. The quadratic function $Q_c$ is a *two-to-one* mapping. In particular, this means that each contour is covered twice by the preceding contour. Remember that every point $z$ has two ancestors.
1. If the sequence of magnitudes $|z_n|$ approaches $\infty$, then it is eventually monotonic, but the first few terms need not be. For instance, choose $z_0$ to be outside the Julia set but very close to the Julia point whose index is $\frac{3}{2}$. Then $z_1$ is very close to the Julia point whose index is $\frac{2}{7}$, and is therefore closer to zero than $z_0$ is.

2. To find the 2-cycle points, solve the equation $z = Q_c^2(z) = (z^2 + c)^2 + c$, which can be rewritten as $z^4 + 2cz^2 - z + c^2 + c = 0$. Because the finite fixed points both satisfy this equation, $z^2 - z + c$ must be a factor:

$$(z^2 - z + c)(z^2 + z + c + 1) = 0$$

It follows that there are two 2-cycle points, given by the quadratic formula as

$$z = \frac{1}{2}(-1 + \sqrt{-3 - 4c}) \quad \text{and} \quad z = \frac{1}{2}(-1 - \sqrt{-3 - 4c}).$$

3. The 3-cycle points of $Q_c$ fit the equation $z = Q_c^3(z)$, which has degree 8. As in the preceding item, however, two of the solutions to this equation are the finite fixed points, leading you to an equation of degree 6 to solve:

$$z^6 + z^5 + (3c + 1)z^4 + (2c + 1)z^3 + (3c^2 + 3c + 1)z^2 + (c + 1)^2z + c^3 + 2c^2 + c + 1 = 0$$

Although the six solutions form two 3-cycles, further factoring seems impossible, even in the special case $z^6 + z^5 + z^4 + z^3 + z^2 + z + 1 = 0$ that occurs when $c = 0$.

4. Because $Q'_c(z) = 2z$, the only critical point is $z = 0$, which is a fixed point only when $c = 0$. (It is possible for $z = 0$ to belong to cycles of length greater than 1, however — consider $c = -1$ as an example. You will soon see that these cycles have the same superattracting property.)

5. If $p$ is an attracting fixed point of $Q_c$, then $|2p| < 1$, or $|p| < \frac{1}{2}$. Because $c = p - p^2$ (see item 7 below), there are infinitely many examples to choose from. For instance, $p = \frac{3}{25} + \frac{4}{25}i$ is an attracting fixed point for $Q_c$ when $c = p - p^2 = \frac{82}{625} + \frac{76}{625}i$.

6. More facts about $Q_c$:
(a) Follows from the identity $Q_c(-w) = Q_c(w)$.
(b) Average the two values in item 5 on page 232.
(c) Attracting fixed points are found inside the circle $|p| = \frac{1}{2}$, and $p = \frac{1}{2}$ is on this circle. Because $p = \frac{1}{2}$ is the midpoint of the segment between the two fixed points, they can not both be inside the circle.
(d) Because of the identity $Q_c(-w) = Q_c(w)$, opposing points have virtually the same orbit and therefore the same destiny.
7. Just rewrite the fixed-point equation \( p = p^2 + c \). Because 1 equals the sum of the fixed points (item 5 on page 20), the equation can also be expressed as \( c = (1 - p)p \), which shows that \( c \) is the *product* of the fixed points (it does not matter which one is called \( p \)).

8. Two simple examples for which \( x = 0 \) is an indifferent fixed point: (a) \( f(x) = \sin x \) and (b) \( f(x) = \sinh x \). The former acts like an incredibly weak attractor; the latter acts like an incredibly weak repeller.

9. The fixed points of \( Q_{-2/3} \) are \( \frac{1}{6}(3 \pm \sqrt{33}) = -0.457427 \) (weakly attracting) and 1.457427 (repelling).

The fixed points of \( Q_{-1} \) are \( \frac{1}{2}(1 \pm \sqrt{5}) = -0.618034 \) (repelling) and 1.618034 (repelling).

The fixed points of \( Q_{-3/4} \) are \(-0.5\) (indifferent) and 1.5 (repelling). If \( z_0 \) is real and near \(-0.5\), then the orbit \( \{z_n\} \) approaches \(-0.5\) very slowly (verify). On the other hand, if \( z_0 = -0.50005 + 0.01i \), then the orbit \( \{z_n\} \) approaches \( \infty \), but it takes a long time to pry itself loose from the fixed point — the first term to leave the circle \( |z| = 2 \) is \( z_{5012} = -1.04188689 + 1.48110404i \).

10. The orbit is described approximately by \( z_n = p + [\frac{1}{2}i]^n(z_0 - p) \), which approaches \( p \) in a geometrically regular fashion: each iteration halves the distance to \( p \) and shifts the angular position by a counterclockwise quarter-turn.

11. Points that are not in the Julia set generate orbits that converge to an attractor; in particular, their orbits do not wander into the Julia set. Ancestors of Julia points are therefore also Julia points, as are successors of Julia points.

The complete basin of attraction (*i.e.*, including all components) of an attractor is another example of a completely invariant set.
1. Any real number can be identified as a limit of dyadic rationals. In particular, the geometric series

\[ \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots = \sum_{n=1}^{\infty} \left( \frac{1}{4} \right)^n \]

has \( \frac{1}{3} \) as its sum. Figure 10 shows the \( \frac{1}{256} \) divisions fairly clearly, so it is not difficult to locate the four-term approximation \( \frac{85}{256} \). The actual location of the point indexed by \( \frac{1}{3} \) is at the vertex of the conspicuous valley just to the left of this approximation. The point indexed by \( \frac{2}{3} \) is directly below, the \( x \)-axis reflection of \( \frac{1}{3} \) in the third quadrant. An approximation formula can be obtained by doubling the geometric series above. These points form the 2-cycle for \( Q_{-5/16} \).

2. One ancestor of \( \frac{1}{3} \) is of course \( \frac{2}{3} \). The other is \( \frac{1}{6} \), which is the sum of the geometric series

\[ \frac{1}{8} + \frac{1}{32} + \frac{1}{128} + \ldots = \frac{1}{2} \sum_{n=1}^{\infty} \left( \frac{1}{4} \right)^n . \]

The \( \frac{1}{512} \) divisions are barely visible, but it is evident that the point you seek is the \( y \)-axis reflection of \( \frac{1}{3} \), just as \( \frac{5}{6} \) is the \( y \)-axis reflection of \( \frac{2}{3} \).

3. This alternating series jumps from one side of \( \frac{2}{3} \) to the other, taking twice as long to converge as the series from item 1:

\[ \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \ldots = 2 \sum_{n=1}^{\infty} \left( \frac{1}{4} \right)^n \]

To see that the two series are equivalent, group the alternating series two terms at a time.

4. A suitably large circle is chosen (\( |z| = 10 \)) to serve as an escape-to-infinity threshold. Once an orbit arrives at a point outside this circle, its position relative to the real axis is noted and a color is assigned — points above the axis are assigned one color, points below are assigned the other. All the previous points along the orbit are then assigned the same color. (Contrast this checkerboard effect with the alternating scheme used on page 22.) Figure 9 shows the threshold circle and a few of its ancestors. The larger the threshold circle is, the more divisions appear in the Julia set close-ups. In fact, by counting the number of divisions, it is possible to estimate the size of the outermost circle.

May 2006 236 Phillips Exeter Academy
1. Points that are dynamically related — i.e., points that belong to the same ancestral tree — will always have neighborhoods that look the same, except for rotation, and minor distortion that becomes imperceptible when the neighborhoods are small enough. This is because the color schemes you see are dynamically defined.

2. The fixed points of $Q_{-5/16}$ are also fixed points for $F$, of course. In addition, the 2-cycle points of $Q_{-5/16}$ are also fixed points for $F$. (In general, a point of period $n$ for $Q_c$ serves as a fixed point for $F = Q_c^n$.) The Chain Rule says that $F'$ is just a product of derivatives:

$$F'(p) = Q'_c(p)Q'_c(Q_c(p)) = 2p \cdot 2Q_c(p) = 4pQ_c(p)$$

Thus $F'(-0.25) = 4(-0.25)^2 = 0.25$, $F'(1.25) = 4(1.25)^2 = 6.25$, and

$$F' \left( \frac{1}{4}(-2 + i\sqrt{7}) \right) = \frac{1}{4}(-2 + i\sqrt{7})(-2 - i\sqrt{7}) = \frac{11}{4} = F' \left( \frac{1}{4}(-2 - i\sqrt{7}) \right)$$

Notice that $F'$ is the same at each member of the 2-cycle — this is to be expected. What is truly noteworthy, however, is that the common value of $F'$ is real. The significance of this will soon become clear. Both $Q_{-5/16}$ and $F$ have the same Julia set.

3. Repelling periodic points always belong to the Julia set. The figure at right shows how the first few terms of the orbit of $z_0 = p + \Delta p$ might look, with $p$ near the lowest dot. Notice that, to obtain $z_{n+1} - p$, each $z_n - p$ is rotated 26.6 clockwise and stretched by the factor 2.24, because $2 - i = \sqrt{5} \cdot \text{cis}(-26.6)$.

4. Divide by 2 and add $\frac{1}{2}$ to obtain the answers $\frac{7}{22}$ and $\frac{9}{11}$.

5. Solving $Q_0^6(z) = z^{64} = z$ gives 64 solutions, two of which are fixed points (period 1), two of which have period 2, six of which have period 3, and 54 of which have period 6. Thus there are nine 6-cycles. Although a precise algebraic solution may be out of the question, the cycle counts are the same for every $Q_c$.

6. When $c = -\frac{3}{4}$, the two members of the cycle merge into one, namely $z = -\frac{1}{2}$. This happens to be one of the fixed points, as could have been anticipated. In fact, the 2-cycle has merged with what used to be the attracting fixed point, which has just become indifferent. Because the 2-cycle is usually found in the Julia set, this suggests that the Julia set has pinched together at the left-hand fixed point. This is indeed the case.

7. Because $pq = 1+c$ (see formulas on page 234), it follows that $c = pq - 1 = -(1+p+p^2)$. 

May 2006
237
Phillips Exeter Academy
Dynamic Fractal Solutions

1. Let $x_0 = 3$ and calculate refinements by the Babylonian rule $x_{n+1} = \frac{1}{2}(x_n + 10x_n^{-1})$. This gives $x_1 = \frac{10}{6}$ and $x_2 = \frac{721}{228} = 3.1622807\ldots$, in error by only 0.000003\ldots

2. Solve the equation $z^{16} = z$ to find that $z_0$ has to be one of the $15^{th}$ roots of 1. One 4-cycle example is cis(24), cis(48), cis(96), cis(192), cis(24), \ldots Another 4-cycle example is cis(72), cis(144), cis(288), cis(216), cis(72), \ldots

3. The formula in item 5 on page 20 shows that the fixed points are different unless $4c = 1$. In other words, $Q_{1/4}(z) = z^2 + \frac{1}{4}$ has a single fixed point, which is $z = \frac{1}{2}$. It is indifferent.

4. The fixed points are $-0.3$ (attracting) and $1.3$ (repelling). The immediate ancestor of 1.3 is $-1.3$, and the immediate ancestors of $-1.3$ are $\pm i\sqrt{0.91} \approx \pm 0.95394i$.

5. The cube roots of $8i = 8\text{cis}(90)$ are $2\text{cis}(30)$, $2\text{cis}(150)$, and $2\text{cis}(270) = -2i$.

6. The circle interior $|z| < 1$ is $Q$-invariant, because $|Q(z)| < 1$ holds whenever $|z| < 1$ does. The circle exterior $1 < |z|$ is $Q$-invariant, because $1 < |Q(z)|$ holds whenever $1 < |z|$ does. The circle $|z| = 1$ is $Q$-invariant, because $|Q(z)| = 1$ holds whenever $|z| = 1$ does. Other examples of $Q$-invariant subsets of the complex plane are the real axis, the positive imaginary axis, and any orbit. On the other hand, the first quadrant is not $Q$-invariant, nor is the imaginary axis, nor is the negative real axis.

7. The fixed points are 0 and $\frac{3}{5}$. Use the derivative $f'(z) = \frac{5}{2}(1 - 2z)$ to find that $f'(0) = \frac{5}{2}$ and $f'(\frac{3}{5}) = -\frac{1}{2}$. Thus 0 is repelling and $\frac{3}{5}$ is attracting.

8. This is a complex square-root finder.
(a) One approach is to load the real and imaginary parts

$$\frac{1}{2} \left( x + \frac{8x + 6y}{x^2 + y^2} \right) \text{ and } \frac{1}{2} \left( y + \frac{6x - 8y}{x^2 + y^2} \right)$$

into a calculator and iterate five or six times. As you have seen, however, this sequence converges to whichever square root of $8 + 6i$ is closer to $i$. Because $8 + 6i = 10\text{cis}(36.870)$, the square roots are $\pm \sqrt{10}\text{cis}(18.435) = \pm(3 + i)$, and so $z_n \to 3 + i$ as $n \to \infty$.
(b) The Julia set for this root-finding system is the line $y = -3x$, so any seed value chosen on this line will do — take $z_0 = 1 - 3i$, for example.

9. The symmetry of the figure suggests a fourth-root finder, and it also suggests that $\text{cis}(45)$ could be one of the roots. Because $\text{cis}(45)^4 = \text{cis}(180) = -1$, it appears that you are looking for the fourth roots of $-1$. The function that produced the picture is

$$N(z) = \frac{1}{4} \left( 3z - \frac{1}{z^2} \right).$$
Dynamic Fractal Solutions

10. The 2-cycle for $Q_c$ is found by solving $(z^2 + c)^2 + c = z$. As in item 2 on page 234, the quadratic formula eventually leads to the answers

$$z = \frac{1}{2}(-1 \pm \sqrt{-3 - 4c}) = \frac{1}{2}(-1 \pm 1.2i) = -0.5 \pm 0.6i.$$

11. Calculate

$$x_{n+1} - 2 = \frac{1}{3} \left( 2x_n + \frac{8}{x_n^2} \right) - 2 = \frac{2(x_n^3 - 3x_n^2 + 4)}{3x_n^2} = \frac{2(x_n - 2)^2(x_n + 1)}{3x_n^2},$$

which shows that $x_{n+1} - 2$ is positive whenever $x_n$ is greater than 2. Because $2 < x_0$, this shows inductively that the sequence $\{x_n\}$ stays to the right of 2. In other words, the interval $2 < x$ is an invariant set.

Now calculate that

$$x_n - x_{n+1} = x_n - \frac{1}{3} \left( 2x_n + \frac{8}{x_n^2} \right) = \frac{x_n^3 - 8}{3x_n^2},$$

which shows that $x_n - x_{n+1}$ is positive whenever $x_n$ is greater than 2. In other words, each term of the sequence $\{x_n\}$ is to the right of the succeeding term, as requested.

As in item 6 on page 205, this essentially shows that the sequence converges to the cube root of 8.

Notice the absence of the first derivative in the above argument. In particular, the fact that $f'(2) = 0$ is of little consequence, for it merely tells you that convergence to $x = 2$ will be rapid if it occurs. Knowing that $f'(x)$ is positive for $2 < x$ is equally inconclusive. To see how inconclusive these two facts are, consider the function $F(x) = 2 + 3(x - 2)^2$, which has some (but not all) of the attributes of the given root-finding function $f$. What happens to the sequence seeded by $x_0 = 3$?

In general, the first derivative is useful in questions about local behavior, but not very useful in questions about global behavior.
Dynamic Fractal Solutions

1. At the very least, every point that is ancestral to the point of index \( \frac{1}{7} \) will have a field line that behaves in the same fashion. These points are dense in the Julia set (as is the case with any ancestral tree). The infinite spiral is in fact typical behavior for a field line.

2. The sum of the series is \( \frac{1}{7} \), which indexes one of the six points of period 3. The sum of any dyadic series is easy to locate in a Julia set, by using the conspicuous ancestors of the repelling fixed point.

page 27

1. The product is \( 4 + 4c \). Because of the Chain Rule, the product is the derivative of \( Q_c^2 \), and it therefore describes the effect of applying \( Q_c \) twice in succession to points near either of the 2-cycle points. When \( c \) is real and \( -1 < c \), then the derivative is real and positive, which tells you that there is no infinitesimal rotation taking place near the 2-cycle points. If \( c \) is real and \( c < -1 \), then the derivative is real and negative, which tells you that there is an infinitesimal 180-degree rotation taking place near the 2-cycle points. If \( c = -1 \), then the derivative is zero, telling you that the 2-cycle is superattracting (see item 5). If \( c = -1 + \frac{1}{8}i \), then \( 4 + 4c = \frac{1}{2}i \), which tells you that the 2-cycle is attracting. Indeed, the 2-cycle is attracting for any \( c \)-value that is within \( \frac{1}{4} \) of \(-1\).

2. Inequalities that confirm a threshold circle:
   (a) If \( 2 < |z| \), then \( 2|z| < |z^2| \). Because \( |z^2 + c| \) can be no smaller than \( |z^2| - |c| \), it follows that
   \[ |z| < |z| + |z| - |c| = 2|z| - |c| < |z^2| - |c| \leq |z^2 + c| = |Q_c(z)|. \]
   (b) The preceding part shows that the circle exterior \( 2 < |z| \) is invariant, and that \( |z_n^2| \) is an increasing sequence. To show that these magnitudes approach infinity, reason as in item 1 on page 233:
   \[ \frac{|z_{m+1}|}{|z_m|} \geq \frac{|z_m^2| - |c|}{|z_m|} = |z_m| - \frac{|c|}{|z_m|} > 2 - \frac{|c|}{|z_m|} = r > 1. \]
   In other words, \( |z_{m+1}| \) is guaranteed to be more than \( r \) times as large as \( |z_m| \).
   Because \( z_{m+1} \) is now known to satisfy the same conditions as \( z_m \) did, you can apply the same reasoning to \( z_{m+1} \) and all its successors. Inductively obtain
   \[ |z_n| > r^{n-m} |z_m| \]
   for all \( n \) greater than \( m \), thereby proving that \( |z_n| \) approaches \( \infty \) monotonically.
3. The fixed points are $\frac{1}{2}(1 \pm \sqrt{5})$, which are approximately 1.618 and $-0.618$. Both are repelling, hence must be part of the Julia set. The 2-cycle is $\{0, -1\}$. Because the derivative of $Q_{-1}^2$ is zero at zero, the 2-cycle is superattracting, hence is not part of the Julia set. Instead, the 2-cycle points are black in the figure.

The new collection of conspicuous points consists of the points where the Julia set pinches the ancestral tree of the left-hand fixed point (which is attracting only when $c$ is close enough to the origin). These pinch points are indexed by the same numbers that index the ancestral tree of the 2-cycle — when the cycle is part of the Julia set, that is. When $c = -1$, the 2-cycle is not part of the Julia set, so the indices $\frac{1}{3}$ and $\frac{2}{3}$ both point to the left-hand fixed point, which is in the Julia set. In general, the indices that represent the pinch points are $\frac{1}{3}, \frac{2}{3}, \frac{1}{6}, \frac{5}{6}, \frac{1}{12}, \ldots$; i.e., all proper, reduced fractions of the form $\frac{k}{2m3}$.

4. This was pointed out in item 1 above.

5. First calculate

$$Q_0(T(z)) = T(z)^2 = \left(\frac{z - m}{z + m}\right)^2$$

and

$$T(N(z)) = \frac{N(z) - m}{N(z) + m} = \frac{1}{2} \left(\frac{z + m^2z^{-1}}{z + m^2z^{-1}} - m\right) = \frac{\left(\frac{z^2 + m^2}{z^2 + m^2} - 2mz\right)}{\left(\frac{z^2 + m^2}{z^2 + m^2} + 2mz\right)} = \frac{(z - m)^2}{(z + m)^2},$$

as desired. The fixed points of the Newton-Raphson system are $m$, $-m$, and $\infty$, which $T$ maps to 0, $\infty$, and 1, respectively; notice that these are the fixed points of the quadratic system. That $T$ is a one-to-one function can be shown by calculating its inverse — see item 1 on the next page. Notice also that $T$ maps the Julia set $|z - m| = |z + m|$ onto the unit circle, which is the Julia set for the quadratic system. In particular, the repelling Newton-Raphson fixed point $\infty$ is mapped by $T$ to the repelling quadratic fixed point 0.

The displayed equations above show that each Newton-Raphson orbit corresponds term-by-term to a quadratic orbit. Moreover, the continuity of $T$ and its inverse guarantee that limiting behavior for one orbit corresponds exactly to limiting behavior for the other.
Dynamic Fractal Solutions

1. Using isomorphism to prove convergence:
(a) Solve $T(z) = w$ for $z$ to find that

$$T^{-1}(w) = m \frac{1 + w}{1 - w}.$$  

(b) $T(\infty) = 1$ and $T^{-1}(\infty) = -m$. For the Newton-Raphson system, $\infty$ is repelling and fixed, so it corresponds to the repelling fixed point 1 of the quadratic system. For the quadratic system, $\infty$ is superattracting, so it corresponds to the superattracting $-m$ of the Newton-Raphson system.
(c) If $m = 2 + i$, then $T^{-1}(i) = -1 + 2i$. In the quadratic system, $i$ is ancestral to $-1$, which is ancestral to $1$. It follows that, in the Newton-Raphson system, $-1 + 2i$ is ancestral to $T^{-1}(-1) = 0$, which is ancestral to $\infty$.
(d) The 2-cycle for $N$ consists of (see item 6 on page 11)

$$T^{-1}(\text{cis}(120)) = (2 + i) \frac{1}{2} + \frac{1}{2} i \sqrt{3} = \frac{-1 + 2i}{\sqrt{3}}$$

and

$$T^{-1}(\text{cis}(240)) = (2 + i) \frac{1}{2} - \frac{1}{2} i \sqrt{3} = \frac{1 - 2i}{\sqrt{3}}.$$  

(e) For any $w_0$ inside the unit circle, the sequence $w_n = w_{n-1}^2$ converges to 0; for any $w_0$ outside the unit circle, the sequence $w_n = w_{n-1}^2$ converges to $\infty$. Now apply $T^{-1}$ to such orbits:

For any $z_0 = T^{-1}(w_0)$, the sequence $z_n = N(z_{n-1})$ converges to $m = T^{-1}(0)$ or to $-m = T^{-1}(\infty)$, depending on whether $w_0$ is closer to $m$ or to $-m$.

2. See Figure 15, which is a magnification centered at the repelling fixed point.
(a) The fixed points are $\frac{1}{2} i$ and $1 - \frac{1}{2} i$, which are indifferent and repelling, respectively.
(b) Because $Q'_c(1 - \frac{1}{2}) = 2 - i$, orbits that originate near the repelling fixed point will be repelled along spiral paths, until they get sufficiently far from the fixed point. For example, the invariant field line that lands at the repelling fixed point (see the figure) is the spiral escape route for all orbits that originate on it. Other spiral escape routes will have the same shape near $1 - \frac{1}{2} i$. In particular, notice the arrangement of pinch points (ancestors of the indifferent fixed point) as they escape from the repelling fixed point.
(c) All dyadic field lines are similar to the field line that lands at 0 — in particular, each one must spiral infinitely many times around before landing, as suggested by Figure 15.
(d) The behavior of $Q_c$ near $\frac{1}{2} i$ is described by $Q_c(z) \approx \frac{1}{2} i + (z - \frac{1}{2} i) i$, because $Q'_c(\frac{1}{2} i)$ is $i$. This linear function simply rotates the plane by a quarter-turn about the center $\frac{1}{2} i$, suggesting that $Q_c$ might have infinitely many 4-cycles. You already know, however, that there are only three 4-cycles (no matter what $c$ is). What actually happens to an orbit near $\frac{1}{2} i$ is that it either drifts slowly toward $\frac{1}{2} i$ or slowly away from $\frac{1}{2} i$ (on its way to $\infty$). The slowness is caused by the indifference of the fixed point.
3. Field lines that approach points of the left-hand 4-cycle must spiral infinitely many times in order to land. On the other hand, the field lines indexed by $\frac{1}{5}$, $\frac{2}{5}$, $\frac{4}{5}$, and $\frac{3}{5}$ do not have to spiral to land. The difference is that the product of the four $z$-values in the left cycle is not real, while the product of the four $z$-values in the right cycle is real (the cycle consists of two pairs of conjugates). Recall that the appearance of the Julia set at a 4-cycle point is determined by the derivative of $Q_c^4$ at any one of the points, which is just 16 times the product of the four $z$-values that make up the cycle. If the product is a positive real number — as is the case with the right cycle — then the cycle produces no net rotational effect.

The third cycle is found by just forming the conjugates of the first cycle, a valid procedure because the Julia set has $x$-axis symmetry. The third 4-cycle is $z_0 = 1.08184 - 0.27724i$, $z_1 = 0.78102 - 0.59986i$, $z_2 = -0.06234 - 0.93700i$, and $z_3 = -1.18659 + 0.11682i$. This cycle requires that its field lines spiral infinitely in order to land — in the sense opposite to the field lines of the first 4-cycle, in fact.

4. For $p$ to be indifferent, $|2p| = 1$, so $p$ must have the form $\frac{1}{2}\text{cis}\theta$. The corresponding values of $c$ may therefore be described by

$$c = p - p^2 = \frac{1}{2}\text{cis}\theta - \frac{1}{4}\text{cis}2\theta,$$

which is equivalent to the parametric form

$$x = \frac{1}{2}\cos\theta - \frac{1}{4}\cos2\theta$$
$$y = \frac{1}{2}\sin\theta - \frac{1}{4}\sin2\theta$$

graphed at right. When $c$ is on this cardioid, $Q_c$ has an indifferent fixed point; when $c$ is inside the cardioid, $Q_c$ has an attracting fixed point.
1. The most prominent dyadic Julia points in Figure 16 are collinear with the 2-cycle point. Their distances from the 2-cycle point are 3.0 mm, 7.5 mm, 20.5 mm, and 56.0 mm. Each separation is about 2.75 times as large as its predecessor.

2. The search for the superattracting 2-cycle \((c = -1)\) also turns up the single superattracting 1-cycle \((c = 0)\). In general, looking for \(n\)-cycles will also turn up \(d\)-cycles whenever \(d\) divides \(n\). Thus the superattracting 3-cycle equation \((c^2 + c)^2 + c = 0\) can be factored into \(c(c^3 + 2c^2 + c + 1) = 0\), which reveals the 1-cycle solution \(c = 0\) (again) as well as three superattracting 3-cycles. With the help of an equation solver, these are found to be

\[
\begin{align*}
c_1 &= -1.7548776662466928 \\
c_2 &= -0.1225611668766536 + 0.7448617666197442i \\
c_3 &= -0.1225611668766536 - 0.7448617666197442i
\end{align*}
\]

Remember that each of these represents a different quadratic polynomial \(Q_c\) and a different Julia set. Two of these are displayed below — \(c_1\) on the left and \(c_2\) on the right.

3. As in item 1, the right-hand fixed point lines up with certain ancestors of the left-hand fixed point, and also with certain ancestors of the right-hand fixed point. Some sequences of measurements are \(\{4.5\ \text{mm}, 14.5\ \text{mm}, 47.0\ \text{mm}\}\), \(\{10.0\ \text{mm}, 33.0\ \text{mm}\}\), \(\{7.5\ \text{mm}, 23.5\ \text{mm}\}\), and \(\{19.5\ \text{mm}, 62.5\ \text{mm}\}\). The derivative multiplier at work here is \(1 + \sqrt{5} = 3.236\).

4. See item 4 on page 32.
Dynamic Fractal Solutions

1. The initial stage has three segments, and each step of the construction replaces one segment by four segments. In particular, notice that the perimeter of each stage is $\frac{4}{3}$ times the perimeter of the preceding stage, implying that the actual snowflake curve has infinite perimeter.

2. The sum of the exterior angles of any polygon is 360. For this to apply to a non-convex polygon, some exterior angles must contribute negatively to the total, namely those where the interior angle exceeds 180. Thus each acute point contributes 120 to the total, and each obtuse point contributes $-60$. Let $x$ and $y$ be the number of acute and obtuse points, respectively. The equations

$$x + y = 3 \cdot 4^n$$
$$120x - 60y = 360$$

can be routinely solved to give $x = 4^n + 2$ and $y = 2 \cdot 4^n - 2$.

3. First of all, notice that $\left(\frac{9}{4}, 0\right)$ is not an obtuse point, for such points occur only at rational coordinates with powers of three in their denominators. To show that that $\left(\frac{9}{4}, 0\right)$ persists, it suffices to show that it is never found in the middle third of any subinterval that arises during the construction, and is therefore never erased. Notice that it is found three fourths of the way from $(0, 0)$ to $(3, 0)$ — to the right of the removed middle third, one fourth of the way from $(2, 0)$ to $(3, 0)$ — to the left of the removed middle third, and then three fourths of the way from $(2, 0)$ to $\left(\frac{7}{3}, 0\right)$ — to the right of the removed middle third. Inductively, it is now clear that this pattern will save $\left(\frac{9}{4}, 0\right)$ from erasure throughout the construction. The preceding shows that $\left(\frac{9}{4}, 0\right)$ can be located by expressing $\frac{9}{4}$ as the sum of the triadic series $3 - \frac{1}{3} - \frac{1}{9} + \frac{1}{27} - \ldots$.

4. This is obvious to anyone who can see the unending parade of tiny equilateral triangles.

5. When a middle-third segment is erased, its indices are linearly stretched over the two segments that take its place. This preserves the indices of any persistent points, of course. It can be proved that this inductive process produces a continuous (but not differentiable) parametrization of the snowflake. This is remarkable, for the snowflake has infinite length.

6. The sequence that begins at $B$, proceeds to the topmost acute point, then to the acute point that is one third of the way to $A$, etc can be described by the geometric series

$$3 + \sqrt{3} \text{cis}(150) + \frac{1}{\sqrt{3}} \text{cis}(210) + \frac{1}{3\sqrt{3}} \text{cis}(270) + \ldots = 3 + \sqrt{3} \text{cis}(150) \sum_{n=0}^{\infty} \left(\frac{\text{cis}(60)}{3}\right)^n,$$

whose sum is $\left(\frac{15}{14}, \frac{3}{14}\sqrt{3}\right)$. Incidentally, this point is indexed by $\frac{13}{15}$, according to the scheme described in item 5 above.
1. The polar equation \( r = \frac{1}{2}(1 - \cos \theta) \) can be converted to Cartesian parametric equations

\[
x = \frac{1}{2}(1 - \cos \theta) \cos \theta \quad \text{and} \quad y = \frac{1}{2}(1 - \cos \theta) \sin \theta.
\]

Shifting right by \( \frac{1}{4} \) leads to the equations

\[
x = \frac{1}{2} \cos \theta - \frac{1}{4}(2 \cos^2 \theta - 1) \quad \text{and} \quad y = \frac{1}{2} \sin \theta - \frac{1}{2} \cos \theta \sin \theta,
\]

which are equivalent to the ones given in item 4 on page 243.

2. See Figure 21. The fixed point is \( p = \frac{1}{2} \text{cis}(120) \), and \( Q'_c(p) = \text{cis}(120) \), which — as a multiplier — acts to simply rotate vectors counterclockwise by 120. This example is analogous to the 4-cycle example shown in Figure 14, differing mainly in the placement of the fixed point and in the number of black components that surround it. This filled-in Julia set is called the fat rabbit.

3. Choosing \( \theta = 240 \) gives another fat rabbit, choosing \( \theta = 270 \) gives the \( x \)-axis mirror image of the 4-cycle example shown in Figure 14, choosing \( \theta = 144 \) gives a 5-cycle example, with its fixed point at \( p = \frac{1}{2} \text{cis}(144) \), and so on. All these examples are indifferent. If \( \theta = 360m/n \) degrees, the left-hand fixed point merges with an \( n \)-cycle. The difficult examples that occur when \( \theta \) is not a rational angle will be appearing soon.

4. This is a parametric description of the examples \( Q_c \) that have indifferent 2-cycles. The value of \( \theta \) describes what sort of pinching is taking place. For example, \( \theta = 90 \) defines an example in which the derivative of \( Q_c^2 \) at either 2-cycle point is \( i \), which says that a four-fold pinch is taking place at each member of the 2-cycle. In other words, the 2-cycle is being pinched by (is coincident with) an 8-cycle, which is on the verge of becoming an attractor, just as the 2-cycle is on the verge of becoming repelling.

5. Figure 20 illustrates the San Marco Julia set. The doubly-indexed Julia points are the ancestors of the 2-cycle. They can also be thought of as the ancestors of the left-hand fixed point. The \( c \)-value marks the tangency between the circle and the cardioid that is discussed in item 7 below.

6. The attractiveness of a cycle \( z_1, z_2, \ldots, z_n \) is determined by the product of derivatives \( 2z_1 \cdot 2z_2 \cdots 2z_n \). If one of the cycle members is zero, then the product is zero, and so the cycle is actually superattracting.
7. The two curves are tangent at \( c = -\frac{3}{4} \), which represents the indifferent case illustrated in item 5 above. These curves outline two of the components of the Mandelbrot set, defined on page 32.

8. The Newton-Raphson examples are exceptional with regard to their critical orbits, because each attracting 1-cycle is placed at a critical point. In other words, superattracting cycles automatically fulfill the promise made by Julia’s theorem. The quadratic examples \( Q_c \) we have seen have the common feature that the origin is completely enclosed by the Julia set, signifying that the critical orbit approaches whatever attractor is contained in the black region.

9. If \( c = -2 \), then the critical orbit is eventually fixed at 2. If \( c = -\frac{3}{4} \), then the critical orbit approaches the indifferent fixed point (1-cycle) \( p = -\frac{1}{2} \). If \( c = -1 \), the critical orbit is the 2-cycle \( \{-1, 0\} \). If \( c = -1 + i \), the critical orbit escapes to infinity. If \( c = i \), the critical orbit is an eventual 2-cycle \( \{-i, -1 + i\} \).

None of the preceding examples fits the description wanders aimlessly in the complex plane. If there were such an example, its wanderings would actually be confined to the disk \(|z| \leq 2 + |c|\), according to item 4 on page 28. Soon, you will encounter a wandering critical orbit, and discover that its behavior is far from aimless.
1. If $0.25 < c$, then the critical orbit escapes to infinity, as suggested by the web diagram shown at right. If $c$ is close to 0.25, however, it can take many iterations for the orbit to squeeze through the narrow opening between the graphs of $y = x$ and $y = x^2 + c$. See page 253 for more.

2. If $2 < |c|$, then $|c| < |c^2 + c|$, because

$$2|c| - |c| < |c^2| - |c| \leq |c^2 + c|.$$  

In other words, $|c| < |Q_c(c)|$. Now apply item 4b on page 28 to see that the critical orbit approaches $\infty$.

3. No matter what $c$ is, $Q^n_c(z) = z$ is a polynomial equation of degree $2^n$, hence it has $2^n$ solutions. Thus any $Q_c$ has a full complement of repelling $n$-cycles, so that its Julia set is infinite. If $c$ is not in the Mandelbrot set, these points form a totally disconnected dust, however.

4. The left-hand fixed point is $\frac{1}{2}(1 - \sqrt{5})$, so the local multiplier is $1 - \sqrt{5} = -1.236\ldots$. This tells you that $Q_{-1}$ acts on this figure by a 180-degree rotation and a radial expansion, so $Q_{-1}^2$ acts by a radial expansion of about $(1 - \sqrt{5})^2 = 1.528$. This agrees with the geometric progression of decorations found in Figure 22, which are found at distances of about 3.0 mm, 4.5 mm, 7.0 mm, 10.5 mm, 16.0 mm, 24.5 mm, and 37.5 mm from the fixed point.

5. Because points in the Julia set are repelling by nature, repeated application of $Q_c$ to nearby points will drive the resulting orbits further away. When this process is run in reverse, however, the Julia set attracts every ancestral tree.

6. Compare Figure 23, which was made with a ceiling of 50 repetitions, with Figure 24, which was made with a ceiling of only 20 repetitions. Notice how all the detail inside the 21st escape contour is lost by using the low ceiling. On the other hand, a high ceiling makes the drawing take more time, because every black point reaches that ceiling. If the additional detail gained by raising the ceiling is invisible, the extra time spent drawing is of course wasted.
Dynamic Fractal Solutions

1. Figure 27 shows this component more clearly. All the $c$-values inside the disk define quadratic systems $Q_c$ that have attracting 4-cycles. Among them, there is one that is superattracting, meaning that 0 belongs to the attracting 4-cycle. This central $c$-value is $0.2822713907669139 + 0.530060617575785253i$.

2. The highest point on the cardioid occurs where the derivative of $y = \frac{1}{2} \sin \theta - \frac{1}{4} \sin 2\theta$ is zero. This leads to the equation $\cos \theta = \cos 2\theta$, whose solutions are $\theta = 0$, $\theta = 120$, and $\theta = 240$. The one you want is $\theta = 120$, which gives $c = \frac{1}{8} (-1 + 3i \sqrt{3})$ as the indifferent example. All $c$-values from within the disk define functions $Q_c$ that have attracting 3-cycles. The central value $c = -0.1225611668766536 + 0.7448617666197442i$ defines the superattracting example. The component is not actually circular, so the terminology center is only suggestive (see page 65).

3. The $c$-value $-1.7548776662466928$ defines a $Q_c$ that has a superattracting 3-cycle, so $c$ must be in the Mandelbrot set somewhere, probably at the center of a component. What is unusual about this component is that it does not bud from the main cardioid, nor from any component that does. Instead, it is found in the largest speck in Figure 25. That speck appears magnified in Figure 28, which suggests that the speck is a miniature version of the complete Mandelbrot set, and that the component you seek is enclosed by another cardioid — this one housing 3-cycle examples instead of 1-cycle examples. It is not a true cardioid, however, and Figures 29, 30, and 31 show that satellites like this one are not exact copies of the the full Mandelbrot set of which they are part.

4. The single fixed point (which is indifferent) is $z = \frac{1}{2}$. Its immediate ancestor is $-\frac{1}{2}$. These points mark the prominent cusps on the real axis. The other cusps mark the rest of the ancestral tree of the fixed point. No dyadic field line has to spiral infinitely often to land. The pure imaginary points are $\pm \frac{1}{2} \sqrt{3}$. The 2-cycle is $-\frac{1}{2} \pm i$. Its field lines land without spiralling, because the product of the 2-cycle points is real. The viewing window for Figure 26 is $3 \times 3$.

5. The equation $Q_c^4(0) = 0$ has degree eight, hence it has eight solutions. One of them defines a 1-cycle, another a 2-cycle, which leaves six superattracting 4-cycles. The equation $Q_c^5(0) = 0$ has degree sixteen, hence it has sixteen solutions. One defines a 1-cycle, leaving fifteen superattracting 5-cycles.
The Mandelbrot set consists of more than just disk-like and cardioid-like components. The whole set is connected by filaments, which can not be seen unless they are highlighted in some fashion. Figures 29, 30, and 31 show the full Mandelbrot set, the 3-cycle satellite centered at $c = -1.754877662466928$, and the 4-cycle satellite centered at $c = -0.1565201668337551 + 1.0322471089228318i$, respectively. The third figure magnifies the largest speck visible near the top of Figure 25. Notice that the images are distinguishable from one another because of the characteristic way that the filaments decorate them. This is why the Mandelbrot set is not self-similar, and why its fractal nature is very different from that of a Julia set.

These figures were drawn using the *Distance-Estimator Method*. As the name suggests, there is a theoretical formula that allows one to calculate, given a point $p$ that is outside the Mandelbrot set, the distance from $p$ to the nearest point of the Mandelbrot set. A disk of that radius therefore does not intersect the Mandelbrot set. Drawing many such disks eventually fills in the exterior of the Mandelbrot set. By making the disks deliberately too small, one ensures that every filament is thickened and thereby left showing. The formula and its derivation are described on pages 62 and 63.
1. The equation $Q^7_c(0) = 0$ is of degree 64, hence there are 64 values of $c$ that satisfy it. Only one of these, $c = 0$, represents a cycle of length less than 7, so there are 63 superattracting 7-cycle centers. Six of the 63 components bud from the main cardioid. The other 57 are satellites. Nine are skewed on the real axis, and the remaining 48 form 24 conjugate pairs.

2. The equation $Q^8_c(0) = 0$ is of degree 128, hence there are 128 values of $c$ that satisfy it. Eight of these represent cycles of length less than 8, so there are 120 superattracting 8-cycle centers. Four of the 120 components bud from the main cardioid, four more bud from buds, and one buds from a bud on a bud. The other 111 are satellites. Fifteen are skewed on the real axis, and the remaining 96 form 48 conjugate pairs.

3. The most obvious common feature is of course the number (5) of limbs that are attached at the fixed point. The most obvious difference is the speed and sense of the spiralling of these limbs. The right-hand example spirals outwards in a counterclockwise sense, the left-hand example spirals clockwise, and the middle example appears to do neither. For most of the examples chosen from this 5-cycle component, the Julia set will spiral out from the fixed point in one sense or the other. The $c$-values for which there is no spiralling form a simple curve that divides the component into a clockwise half and a counterclockwise half. The center of this 5-cycle component is actually in the counterclockwise half.

4. A rational function has the form $R(z) = \frac{P(z)}{Q(z)}$, where $P$ and $Q$ are polynomials. The equation $z = R(R(R(z)))$ can therefore be simplified to a polynomial equation of some (perhaps very high) degree. If the degree is positive, such an equation must have solutions, hence $R$ must have 3-cycles. For such a cycle to be superattracting, one of the members of the cycle must be a critical point of $R$.

5. To find the 1-cycles (fixed points) solve $z = F(z)$; this gives $z = \frac{1}{2}(1 \pm i\sqrt{3})$. Both fixed points are indifferent, because $|z| = 1$ and $F'(z) = z^{-2}$. To find the 2-cycles, solve $z = F(F(z)) \equiv \frac{1}{1-z}$, which gives only the fixed points. To find the 3-cycles, solve $z = F(F(F(z))) \equiv z$, which shows that every non-fixed point is part of a 3-cycle. Some examples:

$$2 \rightarrow \frac{1}{2} \rightarrow -1 \rightarrow 2 \quad i \rightarrow 1 + i \rightarrow \frac{1}{2}(1 + i) \rightarrow i \quad 1 \rightarrow 0 \rightarrow \infty \rightarrow 1$$

Every 3-cycle $z \rightarrow \frac{z-1}{z} \rightarrow \frac{1}{1-z} \rightarrow z$ is indifferent, because the product of its derivative multipliers is

$$\frac{1}{z^2} \left(\frac{z}{z-1}\right)^2 (1-z)^2 = 1.$$
Dynamic Fractal Solutions

5. To find 1-cycles, solve \( z = G(z) \), which gives \( z = \pm i \). Next use \( G'(z) = 2(z + 1)^2 \) to find that \( G'(i) = -i \) and \( G'(-i) = i \), showing both fixed points to be indifferent. To find 2-cycles, solve \( z = G(G(z)) \equiv -\frac{1}{z} \), which gives only the fixed points. To find 3-cycles, solve \( z = G(G(G(z))) \equiv \frac{1 + z}{1 - z} \), which gives only the fixed points. To find 4-cycles, solve \( z = G^4(z) \equiv z \), which shows that any non-fixed point is part of a 4-cycle. Examples:

\[ 2 \rightarrow \frac{1}{3} \rightarrow -\frac{1}{2} \rightarrow -3 \rightarrow 2 \quad 1 \rightarrow 0 \rightarrow -1 \rightarrow \infty \rightarrow 1 \]

The remaining cycles are 4-cycles. Notice that \( z \rightarrow \frac{z - 1}{z + 1} \rightarrow -\frac{1}{z} \rightarrow \frac{1 + z}{1 - z} \rightarrow z \) is indifferent, because the product of its derivative multipliers is

\[ \frac{2}{(1 + z)^2} \cdot \frac{2}{\left(1 + \frac{z - 1}{z + 1}\right)^2} \cdot \frac{2}{\left(1 - \frac{1}{z}\right)^2} \cdot \frac{2}{\left(1 + \frac{1 + z}{1 - z}\right)^2} = 1. \]

Incidentally, \( G \) represents a 90-degree rotation of the complex sphere.

6. As happens in indifferent examples, the Julia set is pinching. Instead of pinching the fixed point (and its ancestors), however, the Julia set is pinching 0 (and its ancestors). Because there is probably no Mandelbrot component attached to the cardioid at this indifferent \( c \)-value (it is indexed by an irrational angle), the slightest outward movement by \( c \) would cause the Julia set to become a disconnected dust of points. It thus seems that the pinch at 0 is preparing the way for the critical orbit to escape to infinity.

7. When \( 0 \leq c \leq 0.25 \), it is clear that the critical orbit stays inside the \( 2p \times 2p \) square, for it approaches the left-hand fixed point monotonically, or else it is constant (\( c = 0 \)). When \( c < 0 \), the issue is simply whether the vertex of the parabola \( y = x^2 + c \) is below the bottom edge of the square, because the graph does not leave the top or the sides of the square, by definition. In other words, you are assured that the critical orbit will stay inside the square whenever \(-p \leq c \), which is equivalent to \(-p \leq p - p^2 \), or \( p^2 \leq 2p \), or \( p \leq 2 \), because \( p \) is positive. To finish the analysis, notice that \( p \leq 2 \) is equivalent to \( 1 + \sqrt{1 - 4c} \leq 4 \), which is equivalent to \( \sqrt{1 - 4c} \leq 3 \), which is equivalent to \( 1 - 4c \leq 9 \), which is equivalent to \(-2 \leq c \).

8. The Chain Rule shows that \( G'(p) \), \( G'(q) \), and \( G'(r) \) all equal \( F'(r)F'(q)F'(p) \), the cumulative local effect of going around the 3-cycle \( p \rightarrow q \rightarrow r \rightarrow p \) once.

9. The web diagram at right shows the steady but slow progress of the critical orbit toward its limiting value of \( \frac{1}{2} \). Because any orbit that starts to the right of \( \frac{1}{2} \) will converge to infinity (no matter how close to \( \frac{1}{2} \) it starts), it is not correct to call \( \frac{1}{2} \) an attractor.
1. Because it is impractical to set such a high ceiling on iterations, many points will be judged stable (left black) long before their instability appears. One is therefore skeptical of points near the border that have been left black.

2. Only four are actually attached to the cardioid, where $\theta = 45, \theta = 135, \theta = 225,$ and $\theta = 315$. See pages 42, 43, and 45 for discussion of the other 8-cycle components.

3. The interior of the unit circle is not closed, for sequences inside the circle can converge to points on the circle. The limit of the sequence $\{\frac{1}{n}\}$ is 0, which is not a number of the form $\frac{1}{n}$, so — lacking its limit — the sequence is not closed. The orbit of zero approaches 1, so the orbit — which does not include 1 — is not a closed set. The ancestral tree of the repelling fixed point is a dense subset of the Julia set, meaning that every Julia point is a limit of points that are ancestral to the repelling fixed point. Because there are many Julia points that do not themselves belong to this ancestral tree, the ancestral tree is not closed. This can in fact be said of any ancestral tree in the Julia set.

4. Closure answers:
   (a) $|z| \leq 1$
   (b) the whole $xy$-plane
   (c) the unit circle
   (d) yes — take any two ancestral trees in a Julia set, or consider the rationals and the irrationals on the real number line
   (e) add the segment $-1 \leq y \leq 1$ on the $y$-axis

5. In those indifferent cases where there is only one attractor (infinity), it is incorrect to think of the Julia set as a curve that separates two (or more) competing basins of attraction.

1. The indices of the fixed point $p$ are $\frac{1}{7}, \frac{2}{7},$ and $\frac{3}{7}$. Its immediate ancestor (which is just $-p$) has indices $\frac{1}{14}, \frac{9}{14},$ and $\frac{11}{14}$. These can be calculated by simply adding $\frac{1}{2}$ to the indices of the fixed point. The field lines must spiral to land, but slowly. To see this, calculate $[Q_c'(p)]^3 = 8p^3 = [1 - \sqrt{1 - 4c}]^3$, where $c = -0.122561 + 0.744862i$. This gives $8p^3 = [-0.55267 + 0.95945i]^3 = [1.1072cis119.94] = 1.357cis59.83$. Thus there is only $-0.17$ degrees of turning for each application of $Q_c^2$. 
Dynamic Fractal Solutions

2. In terms of their dynamic indices, the nine 6-cycles are

\[
\begin{align*}
\frac{1}{63} &\rightarrow \frac{2}{63} \rightarrow \frac{4}{63} \rightarrow \frac{8}{63} \rightarrow \frac{16}{63} \rightarrow \frac{32}{63} \rightarrow \frac{1}{63} \\
\frac{2}{63} &\rightarrow \frac{3}{63} \rightarrow \frac{6}{63} \rightarrow \frac{12}{63} \rightarrow \frac{24}{63} \rightarrow \frac{48}{63} \rightarrow \frac{3}{63} \\
\frac{5}{63} &\rightarrow \frac{10}{63} \rightarrow \frac{20}{63} \rightarrow \frac{40}{63} \rightarrow \frac{17}{63} \rightarrow \frac{34}{63} \rightarrow \frac{5}{63} \\
\frac{7}{63} &\rightarrow \frac{14}{63} \rightarrow \frac{28}{63} \rightarrow \frac{56}{63} \rightarrow \frac{49}{63} \rightarrow \frac{35}{63} \rightarrow \frac{7}{63} \\
\frac{11}{63} &\rightarrow \frac{22}{63} \rightarrow \frac{44}{63} \rightarrow \frac{25}{63} \rightarrow \frac{50}{63} \rightarrow \frac{37}{63} \rightarrow \frac{11}{63} \\
\frac{13}{63} &\rightarrow \frac{26}{63} \rightarrow \frac{52}{63} \rightarrow \frac{41}{63} \rightarrow \frac{19}{63} \rightarrow \frac{38}{63} \rightarrow \frac{13}{63} \\
\frac{15}{63} &\rightarrow \frac{30}{63} \rightarrow \frac{60}{63} \rightarrow \frac{57}{63} \rightarrow \frac{51}{63} \rightarrow \frac{39}{63} \rightarrow \frac{15}{63} \\
\frac{23}{63} &\rightarrow \frac{46}{63} \rightarrow \frac{29}{63} \rightarrow \frac{58}{63} \rightarrow \frac{53}{63} \rightarrow \frac{43}{63} \rightarrow \frac{23}{63} \\
\frac{31}{63} &\rightarrow \frac{62}{63} \rightarrow \frac{61}{63} \rightarrow \frac{59}{63} \rightarrow \frac{55}{63} \rightarrow \frac{47}{63} \rightarrow \frac{31}{63}
\end{align*}
\]

The location of the pinched 3-cycle suggests that one index is between \(\frac{1}{7}\) and \(\frac{1}{7}\), one is between \(\frac{1}{4}\) and \(\frac{9}{7}\), one is between \(\frac{2}{7}\) and \(\frac{9}{8}\), one is between \(\frac{7}{8}\) and \(\frac{3}{11}\), one is between \(\frac{17}{11}\) and \(\frac{1}{7}\). Only the cycle \(\frac{5}{63} \rightarrow \frac{10}{63} \rightarrow \frac{20}{63} \rightarrow \frac{40}{63} \rightarrow \frac{17}{63} \rightarrow \frac{34}{63} \rightarrow \frac{5}{63}\) meets these requirements. In particular, the pinch just below the origin has indices \(\frac{5}{63}\) and \(\frac{40}{63}\), while the pinch just above the origin has the ancestral indices \(\frac{17}{126}\) and \(\frac{73}{126}\).

See Figure 47 for a rabbit that has undergone period-tripling.

3. As noticed in item 2, the 6-cycle \(\frac{5}{63} \rightarrow \frac{10}{63} \rightarrow \frac{20}{63} \rightarrow \frac{40}{63} \rightarrow \frac{17}{63} \rightarrow \frac{34}{63} \rightarrow \frac{5}{63}\) determines the location of the 3-cycle in the Julia set. The lowest term of the 3-cycle is pinched by \(\frac{5}{63}\) and \(\frac{40}{63}\), the highest by \(\frac{10}{63}\) and \(\frac{17}{63}\), and the leftmost by \(\frac{20}{63}\) and \(\frac{34}{63}\).

4. \(Q_{-3/4}\) has fixed points \(-\frac{1}{2}\) and \(\frac{3}{2}\), with multipliers \(-1\) and 3, respectively, and the other ancestor of \(\frac{3}{2}\) is \(-\frac{3}{2}\). \(F\) has fixed points 2 and 0, with multipliers \(-1\) and 3, respectively, and the other ancestor of 0 is 3. This suggests congruence of the Julia sets, where a point \(z\) of the \(F\)-system corresponds to \(\frac{3}{2} - z\) in the \(Q\)-system. Indeed,

\[
Q_{-3/4}\left(\frac{3}{2} - z\right) = \left(\frac{3}{2} - z\right)^2 - \frac{3}{4} = \frac{9}{4} - 3z + z^2 - \frac{3}{4} = \frac{3}{2} - 3z + z^2 = \frac{3}{2} - F(z)
\]

shows that the \(F\)-system is isomorphic to the \(Q\)-system. The isomorphism is an isometry, so the Julia sets are congruent. Except that the center is at \(\frac{3}{2}\), the Julia set for \(F\) looks just like Figure 20.

5. There is no pinching except for the left-hand fixed point and its ancestors, so the component must be attached to the main cardioid. Furthermore, the eleven-fold pinch tells you that the component is indexed by a \(\theta\)-value of the form \(360\left(\frac{m}{11}\right)\). The origin is a point of symmetry for the Julia set, and the average of the two fixed points is \(\frac{1}{2}\), so it is not difficult to find that one unit on the invisible coordinate axes is about 60 mm. This puts \(c\) approximately 19 mm to the right and 2 mm above the origin, which is in the limb adjacent to the torso of the filled-in Julia set. Thus \(\theta = \frac{360}{11}\), from which it is easy to check that the tangency takes place at the nearby indifferent value \(0.31677301317 + 0.04291240989i\).

Figure 60 shows another \textit{dragon} from the same component.
1. When 0 is eventually cyclic, there are two positive integers to be recognized. One is $n$, the length of the eventual cycle. The other is $m$, the number of applications of $Q_c$ that take place before the $n$-cycle is reached. In other words, $0 = z_0, z_1, \ldots, z_m$ are the transient terms, and $z_{m+1} = z_{m+1+n}$.

To say that 0 is \textit{eventually fixed} means that $n = 1$ and that $z_{m+1} = z_{m+2}$ for some (minimal) positive integer $m$.

There is only one solution with $m = 1$, for $z_2 = z_3$ means that $c^2+c = (c^2+c)^2 + c$, which implies that $c^2 = (c^2+c)^2$, or $0 = c^4 + 2c^3$. The solution $c = 0$ actually produces a fixed critical cycle, so only $c = -2$ produces a critical orbit that is fixed after the first term.

There are three solutions with $m = 2$, for $z_3 = z_4$ means that

$$(c^2+c)^2 + c = ((c^2+c)^2 + c)^2 + c,$$

which simplifies to

$$(c^2+c)^2 = ((c^2+c)^2 + c)^2.$$ 

Thus $\pm [c^2+c] = (c^2+c)^2 + c$. The positive sign can be ignored, for it duplicates the case $m = 1$. The other sign leads you — after you divide out the extraneous $c = 0$ — to $c^3 + 2c^2 + 2c + 2 = 0$. The solutions to this cubic are approximately

$$c_1 = -1.5436890126920764$$
$$c_2 = -0.2281554936539618 + 1.1151425080399374i$$
$$c_3 = -0.2281554936539618 - 1.1151425080399374i$$

The first solution is found at a branch point on the real axis. Figures 53 and 54 are centered at this value, the second being a tenfold magnification of the first. Figure 55 shows the Julia set itself for $Q_c$. The resemblance between the Mandelbrot set near $c$ and the corresponding Julia set near $c$ is typical of Misiurewicz points, (which always belong to their Julia sets).

The second solution is the tip of an infinitely spiraling filament (as is its conjugate). Figures 56 and 57 are centered at this value, the second being a tenfold magnification of the first. Figure 58 shows the Julia set itself for $Q_c$. The predictable resemblance between the Mandelbrot set near $c$ and the corresponding Julia set near $c$ may help you locate $c$ in the Julia set (it is near the top of the figure).

There are seven solutions when $m = 3$, for $z_4 = z_5$ means that

$$((c^2+c)^2 + c)^2 + c = (((c^2+c)^2 + c)^2 + c)^2 + c,$$

which can be reduced to

$$c^7 + 4c^6 + 6c^5 + 6c^4 + 6c^3 + 4c^2 + 2c + 2 = 0$$

by reasoning similar to the above.
Dynamic Fractal Solutions

2. The figures suggest that the graph of $R$ is obtained by sliding the graph of $P$ down the line $y = x$. This is indeed the case, for

$$P(x + 1) = (x + 1)^2 - (x + 1) - 2 = x^2 + x - 2 = R(x) + 1.$$ 

Put another way, the systems defined by $P$ and $R$ are isomorphic. The Julia set for $R$ is obtained by shifting the Julia set for $P$ one unit to the left.

3. If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots$ is any polynomial, then

$$R(x) = P\left(x - \frac{a_{n-1}}{na_n}\right) + \frac{a_{n-1}}{na_n}$$

is a dynamically equivalent polynomial, and

$$R(x) = a_n \left(x - \frac{a_{n-1}}{na_n}\right)^n + a_{n-1} \left(x - \frac{a_{n-1}}{na_n}\right)^{n-1} + \ldots$$

has coefficient 0 on $x^{n-1}$. 

May 2006

Phillips Exeter Academy
1. The figures suggest that the graph of $S$ is obtained by moving every point on the graph of $P$ halfway to the origin. This is indeed the case, for

$$P(2x) = (2x)^2 - (2x) - 2 = 2S(x).$$

Put another way, the systems defined by $P$ and $S$ are isomorphic. The Julia set for $S$ is obtained by shrinking the Julia set for $P$ by a factor of 2.

2. If $P(x) = ax^n + bx^{n-1} + \ldots$ is any polynomial, then

$$U(x) = a^{-\frac{1}{n-1}}P\left(x^{\frac{1}{n-1}}\right)$$

is a dynamically equivalent polynomial, and

$$U(x) = a^{-\frac{1}{n-1}}\left[a\left(x^{\frac{1}{n-1}}\right)^n + b\left(x^{\frac{1}{n-1}}\right)^{n-1} + \ldots\right]$$

has coefficient 1 on $x^n$.

3. As shown in item 3 on page 46 and item 2 above, an arbitrary cubic polynomial $P(z) = az^3 + kz^2 + ez + d$ is dynamically equivalent to a polynomial $C(z) = z^3 + mz + b$, namely

$$C(z) = \sqrt{a}\left[P\left(\frac{1}{\sqrt{a}}z - \frac{k}{3a}\right) + \frac{k}{3a}\right].$$

4. There are some arbitrary choices. Try to make $\pm 1$ the superattracting fixed points and 0 the other fixed point of $C(z) = az^3 + bz^2 + cz + d$. To get the fixed points right, you must have $d = 0$, $1 = a + b + c$, and $-1 = -a + b - c$, which imply that $b = 0$ and $a + c = 1$. The requirement $C'(1) = 0$ provides another equation, $3a + c = 0$, which finally leads to $a = -\frac{1}{2}$ and $c = \frac{3}{2}$. Thus $C(z) = \frac{1}{2}(3z - z^3)$ has the desired properties.

Figure 59 shows the filled-in Julia set for this polynomial. There are three basins of attraction, for $\infty$ is always an attracting fixed point when nonlinear polynomials are iterated. The most conspicuous points in the figure are ancestral to the repelling fixed point at 0. Each such point looks locally like 0, bordered by all three basins (twice by the infinite basin). The extreme $x$-intercepts $\pm \sqrt{5}$ — which form a 2-cycle — are also conspicuous, as are their ancestors. There are also two more inconspicuous 2-cycles, $\frac{1}{2}(\sqrt{7} + i) \leftrightarrow \frac{1}{2}(\sqrt{7} - i)$ and $\frac{1}{2}(-\sqrt{7} + i) \leftrightarrow \frac{1}{2}(-\sqrt{7} - i)$, as well as infinitely many other repelling cycles in the Julia set.
4. Although there are many other possible cubics (for example, $F(z) = 3z^2 - 2z^3$ has superattracting fixed points at 0 and 1, and a repelling fixed point at $\frac{1}{2}$), it is not difficult to show that all examples are linearly isomorphic, with the repelling fixed point always midway between the superattractors. The picture on page 257 is essentially the only one.

5. The graphs intersect where $x = \frac{1}{2}(1 \pm \sqrt{5})$, the fixed points of $Q_{-1}$. They also intersect where $x = 0$ and where $x = -1$, which form the 2-cycle for $Q_{-1}$. The slopes at the 2-cycle intersections are both 0, because the 2-cycle is superattracting.

6. The case $c = 0$ is representative of what happens in general. Given an index $\theta$ between 0 and 1, you obtain the sequence

$$\frac{\theta}{2}, \frac{\theta}{4} + \frac{1}{2}, \frac{\theta}{8} + \frac{1}{4}, \frac{\theta}{16} + \frac{5}{8}, \frac{\theta}{32} + \frac{5}{16}, \frac{\theta}{64} + \frac{21}{32}, \ldots$$

which approaches the 2-cycle $\frac{1}{3} \leftrightarrow \frac{2}{3}$. Instead of filling the Julia set, the sequence clusters around just two of its points.

7. Notice that $z = 3$ is the fixed point of $L$, and that the displacement vectors $z - 3$ are turned counterclockwise by 90 and halved. For any $z_0 \neq 3$, the orbit $\{z_n\}$ approaches 3 along a geometric spiral.

8. Because $F(\phi(p)) = \phi(G(p)) = \phi(p)$, it follows that $\phi(p)$ is a fixed point of $F$. Apply the Chain Rule to $F(\phi(z)) = \phi(G(z))$ to obtain

$$F'(\phi(z))\phi'(z) = \phi'(G(z))G'(z),$$

then substitute $z = p$ to obtain

$$F'(\phi(p))\phi'(p) = \phi'(G(p))G'(p) = \phi'(p)G'(p).$$

Now divide by $\phi'(p)$ to obtain the required $F'(\phi(p)) = G'(p)$. The division step is legitimate, provided that $\phi'(p)$ is not zero. This is an automatic property of differentiable functions with differentiable inverses, however.

9. The given $c$-values are taken from different Mandelbrot satellites on the real axis. Between these two values lie infinitely many more satellites, with all possible periods represented.
1. All the intersections in the figures signify fixed points of $Q_c^3$. Two of them actually signify the fixed points of $Q_c$, present in all three figures (though slightly different from one to the next, because $c$ is changing). In the third figure, the other six intersections come from the two 3-cycles of $Q_c$, which both happen to be real for $c = -1.76$. The three points of any one 3-cycle all have the same multiplier, which is therefore the slope of three intersections in the figure. The 100-fold magnification allows you to estimate those slopes. One is about $-0.5$, the other about $2$. Thus one of the 3-cycles is attracting and the other is repelling.

The figure at right shows the square box determined by the upper intersection in the 100-fold magnification on page 46. As remarked in item 7 on page 36, this box traps any part of a web diagram that enters it (as do two other companion boxes elsewhere). In particular, these boxes serve to keep the critical orbit from approaching infinity, and this will hold true even if $c$ is varied slightly (including imaginary values, which can not be graphed here). Although the figure at right does not show the graph of an actual quadratic, these allowed variations in $c$ still form a small Mandelbrot-like configuration, whose associated Julia sets look just like the ones you have already seen (except that they are smaller and are pasted into dendrites). This accounts for the presence of the large satellite on the real axis, whose cardioid is centered at $c = -1.7548776662466928$.

Such satellites are ready to blossom whenever the graph of $Q_c^n$ happens to meet the graph of $I(z) = z$ tangentially. The $c$-value that creates the tangential intersection is the cusp of the satellite cardioid, where a connecting filament is attached. Suitably small variations in $c$ (but not in $n$) cause the Julia sets of $Q_c$ to run through the entire Mandelbrot catalogue. Because there are actually $n$ synchronized tangential intersections, the base attractor of the cardioid is an $n$-cycle.

2. Under the conditions stated, $Q_c$ has two real fixed points, which produce the visible intersections with the line $y = x$. (For other values of $c$, the fixed points will be non-real.)

3. Any intersection signifies a fixed point of $Q_c^5$, hence either a 1-cycle or a 5-cycle of $Q_c$. If there are more than two intersections, then this signifies the presence of a real 5-cycle, and there must be at least seven intersections. Five of them belong to a 5-cycle, which has the same multiplier at all of its points, hence the corresponding intersections of $y = Q_c^5(x)$ with $y = x$ all have the same slope.
Dynamic Fractal Solutions

page 46

4. The results of two thousand iterations on the seed value $z_0 = 1$ are shown at right. Although the choice of function was random at each step, this can be thought of as an orbit, and one gets the impression that the orbit seems to be converging to a strange attractor — namely, a quadrilateral (perhaps even a rectangle) that has had some stripes removed from it. One wonders whether it is possible to determine the corners of this quadrilateral. One also wonders where the points $z = 1$ and $z = -1$ are in the figure, for these are the fixed points of the two linear functions $F$ and $G$, respectively.

In fact, the attractor does have a rectangular outline, with corners at $2.6, -1 + 2.4i, -2.6,$ and $1 - 2.4i$. These are the fixed points of the linear functions

$$F(F(G(G(z)))) = \frac{1}{81}(169 + 16z),$$
$$G(F(F(G(z)))) = \frac{1}{81}(-65 + 156i + 16z),$$
$$G(G(F(F(z)))) = \frac{1}{81}(-169 + 16z), \text{ and}$$
$$F(G(G(F(z)))) = \frac{1}{81}(65 - 156i + 16z),$$

respectively. Because the other conspicuous corners on the orbit are dynamically related to these four corners, it is possible to calculate them, too.
1. Compare coefficients of \(z\) and \(z^2\) in
\[
3(mz + b) - 3(mz + b)^2 = m \left( z^2 - \frac{3}{4} \right) + b
\]
to find that \(m = -\frac{1}{3}\) and \(b = \frac{1}{2}\).

2. The Julia set looks exactly like Figure 20. The only differences are invisible: The repelling fixed point is 0, at the extreme left of the Julia set, and the indifferent fixed point is \(\frac{2}{3}\), the first pinch point to the right of the center, which is \(\frac{1}{2}\). The extreme right of the Julia set is 1, an ancestor of 0.

3. The linear change of variables \(w = mz + b\) is uniquely determined by \(a\), \(k\), and \(d\), namely
\[
m = \frac{1}{a} \quad \text{and} \quad b = -\frac{k}{2a},
\]
and so is the parameter value
\[
c = ad + \frac{k}{2} - \frac{k^2}{4}.
\]

4. To find the square roots \(u + vi\) of a complex number \(x + yi\) by direct algebraic computation, it is necessary to solve the simultaneous equations
\[
x = u^2 - v^2
\]
\[
y = 2uv
\]
for \(u\) and \(v\). One obtains eventually
\[
u + vi = \pm \sqrt{\frac{x + \sqrt{x^2 + y^2}}{2}} \pm i \sqrt{\frac{-x + \sqrt{x^2 + y^2}}{2}},
\]
where the signs are chosen to agree when \(y\) is positive, and to disagree otherwise.

5. In order for \(Q_c\) to be dynamically equivalent to \(Q_e\), it is necessary that the fixed points correspond, and that their multipliers be the same (item 8 on page 45). Because the multipliers have the form \(2p\), where \(p\) is a fixed point, it follows that \(Q_c\) and \(Q_e\) must have the same fixed points. Thus \(c = e\), because the equation \(c = p - p^2 = e\) determines the parameter values.

6. The only quadratic example is \(P(z) = -\frac{1 + i}{2} z^2 + \frac{-1 + 3i}{2} z + 1\). The 3-cycle is repelling, because \(P'(0)P'(1)P'(i) = \frac{5}{4}(1 - i)\).
Dynamic Fractal Solutions

May 2006

page 50

1. In order for \( F'(z) = Q'_c(Q_c(z))Q'_c(z) \) to be zero, one of the factors must be zero, which means that either \( z \) is a critical point of \( Q_c \), or else \( z \) is an immediate ancestor of a critical point. In other words, either \( z = 0 \) or \( z = \pm \sqrt{-c} \). The same reasoning applies to \( G'(z) = Q'_c(Q_c(Q_c(z)))Q'_c(Q_c(z))Q'_c(z) \); either \( z \) is critical, or it is an immediate ancestor of a critical point, or an immediate ancestor of an immediate ancestor. In other words, \( z = 0 \) or \( z = \pm \sqrt{-c} \), or \( z = \pm \sqrt{-c} \pm \sqrt{-c} \), seven values in all.

2. Going back four generations gives 1, \(-1\), \( \sqrt{2} \), \(-\sqrt{2} \), \( \sqrt{1+\sqrt{2}} \), \(-\sqrt{1+\sqrt{2}} \), \( \sqrt{1+\sqrt{1+\sqrt{2}}} \), \(-\sqrt{1+\sqrt{1+\sqrt{2}}} \), and so on. Two of these are not real. The ancestral tree approaches the Julia set from within. In particular, the sequence

\[
0, 1, \sqrt{2}, \sqrt{1+\sqrt{2}}, \sqrt{1+\sqrt{1+\sqrt{2}}}, \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{2}}}}, \ldots
\]

approaches the Julia point \( \frac{1}{2}(1+\sqrt{5}) \).

3. Consider \( P(z) = (z-1)^2(1-kz) \), where \( k \) is a constant to be determined. Notice that \( P(1) = 0 \) and \( P(0) = 1 \), so \( 0 \leftrightarrow 1 \) is a 2-cycle. Because \( P'(z) = (z-1)(k+2-3kz) \), the 2-cycle is superattracting. The other critical point is \( z = \frac{k+2}{3k} \); the goal is to choose \( k \) so that this critical point is fixed:

\[
\frac{k+2}{3k} = \left( \frac{2-2k}{3k} \right)^2 \left( \frac{1-k}{3} \right)
\]

\[
k+2 = \frac{4(1-k)^3}{9k}
\]

\[
4k^3 - 3k^2 + 30k - 4 = 0
\]

The real solution to this cubic is \( k = 0.134824 \ldots \), so \( k = \frac{2}{15} = 0.1333 \ldots \) is close to perfect. In other words, \( P(z) = \frac{1}{15}(z-1)^2(15 - 2z) \) has a superattracting 2-cycle and probably a strongly attracting fixed point. Indeed, one fixed point is \( z = 5.41834591745 \ldots \), and \( P'(z) = \frac{2}{15}(z-1)(16 - 3z) = -0.150246006 \ldots \)

The filled-in Julia set for \( P \) is shown in Figure 73. The light gray regions are the attractive basin for the fixed point, the black regions collect orbits for the superattracting 2-cycle, and the attractive basin for \( \infty \) is banded with white and dark gray.

4. It is impossible. Because a cubic polynomial only has two finite critical points, it can have only two finite attractors, each of which must attract a critical orbit (see item 8 on page 31).
5. Because \( C'_k(z) = kz(4 - 3z) \), \( z = 0 \) and \( z = \frac{4}{3} \) are critical points; \( z = 0 \) is also fixed, hence it is superattracting. To find other fixed points, solve \( kz^2(2 - z) = z \); one obtains

\[
z = 0 \text{ and } z = 1 \pm \sqrt{1 - \frac{1}{k}}.
\]

There will be three real fixed points when \( k < 0 \) or when \( 1 < k \). For there to be another superattracting fixed point, the critical point \( z = \frac{4}{3} \) must be fixed; this happens when \( k = \frac{9}{8} \) (see the central figure below). To find indifferent fixed points, one must solve the equation

\[
1 = \left| C'_k \left( 1 \pm \sqrt{1 - \frac{1}{k}} \right) \right|.
\]

The two solutions: \( k = 1 \) makes \( z = 1 \) indifferent (see left-hand figure), and \( k = \frac{4}{3} \) makes \( z = \frac{3}{2} \) indifferent (see right-hand figure).

The right-hand figure shows a period-doubling bifurcation about to occur, as \( k \) increases beyond \( \frac{4}{3} \); for instance, there is an attracting 2-cycle when \( k \) equals 1.385. The left-hand figure shows a tangent-line bifurcation about to occur, as \( k \) increases beyond 1 — as if from nowhere, a fixed point appears and then splits into two fixed points, one attracting and the other repelling.

Another way of finding an attracting 2-cycle is to look specifically for the superattracting one. In other words, find the \( k \)-value that makes \( x = \frac{4}{3} \) part of a 2-cycle. Calculate

\[
C_k^2 \left( \frac{4}{3} \right) = C_k \left( \frac{32k}{27} \right) = \frac{1024k^3}{729} \left( 2 - \frac{32k}{27} \right),
\]

then set this equal to \( \frac{4}{3} \). Because \( k = \frac{9}{8} \) is a known solution (why?), the resulting equation, \( 8192k^4 - 13824k^3 + 6561 = 0 \), can be reduced to \( 1024k^3 - 576k^2 - 648k - 729 = 0 \). Setting \( k = \frac{9}{8} \) transforms this cubic equation into \( 2u^3 - u^2 - u - 1 = 0 \). This yields \( k = 1.38797091959429 \ldots \)
6. \( L(z) = \frac{1}{2}iz + i - 1 \) is the only answer. Notice that there were three conditions given, any one of which was redundant (while being consistent with the others). The fixed point (rotation center) is \( \frac{1}{3}(2i - 6) \).

7. Any orbit will approach \( c \) along a spiral path. In going from one term to the next, each displacement vector \( z - c \) is shortened by the factor of \( r \) and turned counterclockwise through the angle \( \theta \).

8. Let \( p = 0 \) be the fixed point of \( F \), \( q = 2 \) be the fixed point of \( G \), and \( r = 1 + 2i \) be the fixed point of \( H \). If \( z_0 \) is any seed point within the triangle formed by \( p, q, \) and \( r \), then the entire orbit of \( z_0 \) will lie within this triangle. If \( z \) is any point on the segment joining \( q \) to \( r \), then the next point in the orbit will either lie on this line or else on the midline that is parallel to it; similar remarks apply to the other two sides. In fact, similar remarks apply to any of the midlines in the diagram below, which was produced by two thousand random applications of functions \( F, G, \) and \( H \).
That $N(p) = p$ is clear. Notice also that the roots of $E$ are the only fixed points of $N$. Calculate $N'(z) = \frac{E(z)E''(z)}{E'(z)^2}$ to see that these roots are also critical points of $N$.

2. This follows from the preceding calculation of $N'$. Notice also that multiple roots of $E$ (when $E(z)$ and $E'(z)$ are zero simultaneously) cause trouble for the Newton-Raphson method.

3. The $k^{th}$ roots of $b$ are found by the function $N(z) = \frac{1}{k} \left( (k-1)z + \frac{b}{z^k-1} \right)$.

4. The function $N(z) = \frac{2z^3 + 2}{3z^2 + 1}$ solves $z^3 + z = 2$.

5. The function $N(z) = \frac{2z^3}{3z^2 - 5}$ finds the roots of $5z = z^3$. The Julia set of this root finder is shown in Figure 74. The Julia points closest to the origin make up the 2-cycle \{1, -1\}. The immediate ancestors of $z = 1$ are $z = -1$ and $z = \frac{1}{4}(5 \pm i\sqrt{15})$; the immediate ancestors of $z = -1$ are $z = 1$ and $z = \frac{1}{4}(-5 \pm i\sqrt{15})$. Find these in the figure.

6. The equation $4z^3 - (c^2 + 3)z + c^2 - 1 = 0$ has $z = 1$, $z = \frac{1}{2}(-1 + c)$, and $z = \frac{1}{2}(-1 - c)$ for its roots, each of which is superattracting when the root-finding function

$$N(z) = \frac{8z^3 + 1 - c^2}{12z^2 - c^2 - 3}$$

is applied. As noted in item 2 above, the only critical point whose destination is unknown is $z = 0$. (This will be true for any cubic equation the sum of whose roots is zero.) For most cubics, this extra critical orbit simply converges to one of the three roots. Careful selection of the parameter $c$ can produce examples for which an attracting $k$-cycle ($1 < k$) materializes, however. The given $c$-value is such an example ($k = 9$). The Julia set for $N$ separates four basins of attraction, the fourth consisting of those seed values that, like zero, are attracted to the 9-cycle. The biggest surprise is that the components of this fourth domain look just like a familiar quadratic domain of attraction. See Figure 75.
1. In the following list of answers, the notation $U \rightarrow V$ signifies that the region $U$ is transformed (point by point) onto the region $V$:

$$A \rightarrow B \rightarrow C \rightarrow D \rightarrow E \rightarrow C$$

$$C_1 \rightarrow D$$

$$C_2 \rightarrow D$$

$$F \rightarrow H$$

$$G \rightarrow B$$

$$H \rightarrow E$$

$$K \rightarrow H$$

Notice that opposites (for example, $F$ and $K$) always have the same destination ($H$), whether they be points or symmetrically placed congruent blobs. Also notice the special status of the critical region $C$, so called because it contains the critical point 0. Unlike the other regions, which are transformed in a 1-to-1 fashion onto their images, the critical region is transformed in a 2-to-1 fashion onto $D$. Each of the halves ($C_1$ and $C_2$) is transformed 1-to-1 onto $D$.

2. All the lower-case letters are dyadic fractions:

$$p = \frac{1}{8}, \quad q = \frac{1}{4}, \quad r = \frac{1}{2}, \quad s = \frac{9}{16}, \quad t = \frac{19}{32}, \quad u = \frac{5}{8}, \quad v = \frac{3}{4}$$

Notice that opposites (such as $q$ and $v$, or $p$ and $u$) have field indices that differ by $\frac{1}{2}$.

3. This point is ancestral to the pinch point common to $B$, $C$, and $H$, which is ancestral to the left-hand fixed point (common to $C$, $D$, and $E$). Let $U \lor V$ denote the index of the field line that approaches a pinch point shared by regions $U$ and $V$. Because the left-hand fixed point is now where a 3-cycle used to be (in the Julia set), it is not difficult to see that $C \lor D = \frac{1}{7}$, $D \lor E = \frac{2}{7}$, and $E \lor C = \frac{4}{7}$. It follows that $B \lor C = \frac{1}{14}$ and $B \lor A = \frac{1}{28}$. Because $B \lor A$ and $E \lor G$ are opposites, you see that $E \lor G = \frac{1}{28} + \frac{1}{2} = \frac{15}{28}$.
1. The root-finding function is (page 6)

\[ N(z) = z - \frac{z^3 - 3z^2 + 2z + 10i}{3z^2 - 6z + 2} \]

\[ = \frac{2z^3 - 3z^2 - 10i}{3z^2 - 6z + 2} \]

Now calculate

\[ U(N_c(z)) = 2i \frac{8z^3 + 1 - c^2}{12z^2 - c^2 - 3} + 1 \]

\[ = \frac{16iz^3 + 12z^2 + 2i - 2ic^2 - c^2 - 3}{12z^2 - c^2 - 3} \]

and

\[ N(U(z)) = \frac{2(2iz + 1)^3 - 3(2iz + 1)^2 - 10i}{3(2iz + 1)^2 - 6(2iz + 1) + 2} \]

\[ = \frac{16iz^3 + 12z^2 + 1 + 10i}{12z^2 + 1} \]

To make \( U(N_c(z)) \) equivalent to \( N(U(z)) \), it is necessary (and sufficient) to set \( c = 2i \).

Thus the polynomial equations

\[ z^3 - 3z^2 + 2z + 10i = 0 \]

and

\[ 4z^3 + z - 5 = 0 \]

exhibit the same dynamic behavior when Newton’s Method is applied to them.
1. Stage (c) introduces eight great-great grandparents into the following list of ancestors of $p$:

\[ -\sqrt{2.31 + \sqrt{2.31 - 0.21}} \approx -1.9934413 \]
\[ -\sqrt{2.31 + \sqrt{2.31 - 0.21}} \approx -1.9159299 \]
\[ -\sqrt{2.31 + \sqrt{2.31 - 0.21}} \approx -1.6638082 \]
\[ -\sqrt{2.31 - 0.21} \approx -1.3607874 \]
\[ -\sqrt{2.31 - 2.31 - 0.21} \approx -0.9742754 \]
\[ -\sqrt{2.31 - 2.31 + 0.21} \approx -0.8038606 \]
\[ -\sqrt{2.31 + 0.21} \approx 0.4582576 \]
\[ \sqrt{2.31 - 0.21} \approx 0.4582576 \]
\[ \sqrt{2.31 - 2.31 - 0.21} \approx 0.8038606 \]
\[ \sqrt{2.31 - 2.31 + 0.21} \approx 0.9742754 \]
\[ \sqrt{2.31 + 0.21} \approx 1.3607874 \]
\[ \sqrt{2.31 + 2.31 + 0.21} \approx 1.6638082 \]
\[ \sqrt{2.31 + 2.31 - 0.21} \approx 1.9159299 \]
\[ \sqrt{2.31 + 2.31 + 0.21} \approx 1.9934413 \]

Successive pairs are the endpoints of removed intervals. In general, if $a < b$ are the endpoints of a removed interval, then the next step will remove the flanking intervals $\sqrt{2.31 - b} < \sqrt{2.31 - a}$ and $\sqrt{2.31 + a} < \sqrt{2.31 + b}$, as well as their opposites.

2. $2^n$ pieces remain.

3. One expects the sum of this infinite series

\[ 2\sqrt{0.21} + 2 \left( \sqrt{2.31 + \sqrt{0.21}} - \sqrt{2.31 - \sqrt{0.21}} \right) + \cdots \]

... to be $4.2 = 2p$.
1. The equation $c = p - p^2$ (item 7 on page 23) shows that $c = -\frac{4}{9}$. The imaginary intercepts are roots of $z^2 - \frac{4}{9} = -\frac{4}{3}$, or $z = \pm\frac{1}{3}\sqrt{8}$.

2. When $c$ belongs to the 3-cycle cardioid-like component shown on page 250, $Q_c$ has an attracting 3-cycle. When $c = -1.7548776662466928$, the 3-cycle is superattracting. There are infinitely many other real examples nearby.

3. For $z$ to be part of a 6-cycle means that $z = Q^6_0(z) = z^{64}$, or $z^{63} = 1$, so any number of the form $\text{cis}(360n/63)$ will do, as long as $n$ is between 0 and 63 and not divisible by 3.

4. For $F(z) = az^2 + bz + c$ to have $z = 0$ as a fixed point means that $c = 0$. For $z = 2$ to be a superattracting fixed point means that $4a + b = 2$ and $4a + b = 0$. Thus $F(z) = 2z - \frac{1}{9}z^2$. Because 0 is a repelling fixed point, the Julia set is the circle of radius 2 centered at 2. As shown on page 50, this example is dynamically equivalent to $Q_0$.

5. One label is $\frac{19}{64}$; the other ancestor is diametrically opposite, with label $\frac{19}{64} + \frac{1}{2} = \frac{51}{64}$.

6. Repeated label-doubling (throwing away integer parts) must lead to the label 0, thus eventually-fixed labels are dyadic fractions (whose denominators are $2^n$).

7. Calculate $z_1 = L(0) = b$, $z_2 = L(b) = mb + b$, and $z_3 = L(mb + b) = m^2b + mb + b$. The limit of this sequence is the sum of the infinite geometric series

$$b + mb + m^2b + \ldots = \frac{b}{1 - m}.$$ 

Notice that this is the fixed point of $L$. The series converges because $|m| < 1$.

8. The equation-solving function is

$$N(z) = z - Bz - z^3 = \frac{2z^3}{3z^2 - B};$$

so $N(N(3)) = 3$ leads to

$$N\left(\frac{54}{27 - B}\right) = 3,$$

then to

$$2\left(\frac{54}{27 - B}\right)^3 = 9\left(\frac{54}{27 - B}\right)^2 - 3B,$$

and finally to

$$B^4 - 81B^3 + 2187B^2 - 28431B + 131220 = (B - 9)(B + 45)(B^2 - 27B + 324) = 0.$$ 

The process has $3 \leftrightarrow -3$ as a 2-cycle when $B = 45$. What about $B = 9$?
There are six, attached at \( \theta = 40, \theta = 80, \theta = 160, \theta = 200, \theta = 280, \) and \( \theta = 320 \) (see the cardioid equation in item 4 on page 243). The equation \( Q_8^c(0) = 0 \) has 256 roots in all, including the 1-cycle point and the three 3-cycle points. Thus 246 of the 9-cycle components are not attached directly to the main cardioid, although four of them are period-tripling disks attached to the 3-cycle disks.

Infinity is a repelling fixed point, thus it is a member of the Julia set.

The labels \( \frac{1}{4} \) and \( \frac{3}{4} \) are given to the origin — the ancestor of \( z = -2 \), which gets only the label \( \frac{1}{2} \). Labels \( \frac{1}{8} \) and \( \frac{7}{8} \) are given to \( z = \sqrt{2} \), one of the ancestors of the origin. The other ancestor is \( z = -\sqrt{2} \), whose labels are \( \frac{3}{8} \) and \( \frac{5}{8} \). The labels \( \frac{1}{3} \) and \( \frac{2}{3} \) are both given to the interior fixed point at \( z = -1 \). Notice that these labels are not applied to the 2-cycle points, which get labels \( \frac{1}{5}, \frac{2}{5}, \frac{4}{5}, \) and \( \frac{3}{5} \) (see the next item).

The identity

\[
(z + \frac{1}{z})^2 - 2 = z^2 + \frac{1}{z^2}
\]

demonstrates the isomorphism, which takes the form \( \phi(Q_0(z)) = Q_{-2}(\phi(z)) \). Notice that \( \phi \) associates points of the \( Q_0 \)-system to points of the \( Q_{-2} \)-system; in particular, points that are on the unit circle are associated to points of the real segment \(-2 \leq x \leq 2\), the explicit correspondence matching \( \text{cis} \theta \) with \( 2 \cos \theta \). In other words, the label \( k \) is matched with the real point \( x = 2 \cos(360k) \). For example, \( \frac{2}{5} \) is matched with \( x = 2 \cos 144 \), and \( \frac{1}{5} \) is matched with the fixed point \( x = 2 \cos 120 = -1 \). See Figure 84.
1. Results for $Q_{-1}$:

(a) The 2-cycle is found in components $C$ and $B$.

(b) In the following list of answers, the notation $U \rightarrow V$ signifies that the region $U$ is transformed (point by point) onto the region $V$:

$$E \rightarrow D \rightarrow C \rightarrow B \rightarrow C$$
$$A \rightarrow D$$
$$F \rightarrow A$$
$$G \rightarrow A$$

Notice that opposites (for example, $A$ and $E$) always have the same destination ($D$). Also notice the special status of the critical region $C$, so called because it contains the critical point 0. Unlike the other regions, which are transformed in a 1-to-1 fashion onto their images, the critical region is transformed in a 2-to-1 fashion onto $B$.

(c) The field lines that pass between $C$ and $F$ are mapped to the field lines that pass between $A$ and $B$, which are mapped to the field lines that pass between $D$ and $C$, which are mapped to the field lines that pass between $C$ and $B$, which are labelled $13$ and $23$. Thus the field lines that pass between $D$ and $C$ are labelled $16$ and $56$, the field lines that pass between $A$ and $B$ are labelled $712$ and $512$, and the field lines that pass between $C$ and $F$ are labelled $724$ and $524$.

2. Each application of $F$ turns the displacement vector 30 clockwise and doubles its length. Thus the dot $F^4(q)$ should be about 2 inches away from $p$, with its vector pointing 30 into the third quadrant.

3. The mappings:

$$a \rightarrow b \rightarrow d \rightarrow d \quad g \rightarrow f \rightarrow h \rightarrow h \quad x \rightarrow n \rightarrow b \quad e \rightarrow c \rightarrow f$$

4. The repelling fixed point at infinity gets the home label 0. The origin (the immediate ancestor of $\infty$) gets the label $\frac{1}{2}$. The origin has two ancestors in the white boundary, which get labels $\frac{1}{4}$ and $\frac{3}{4}$. The former is assigned to the boundary point shared by $c$ and $h$, and the latter is assigned to the point shared by $a$ and $b$, for the labelling proceeds in a counterclockwise sense. The point shared by regions $e$ and $h$ is an immediate ancestor of the point shared by $c$ and $h$; it gets the label $\frac{1}{8}$. Where are the other ancestors?

5. There are infinitely many cubic examples

$$F(z) = az^3 + bz^2 + cz + d.$$ 

If $c = 0$, then $F(z) = \frac{1}{4}(4 + 9z^2 - 5z^3)$, which has $0 \rightarrow 1 \rightarrow 2 \rightarrow 0$ as a superattracting cycle, because 0 is a critical point of $F$. Figure 85 shows the filled-in Julia set for this polynomial.
1. If $a$ and $b$ are different complex numbers for which $F(a) = F(b)$, then $a - \frac{1}{2}a^2 = b - \frac{1}{2}b^2$, which can be simplified to $2(a - b) = a^2 - b^2$, and then to $2 = a + b$, because $a - b$ is not zero. It follows that either $a$ or $b$ must be outside the unit disk.

2. Properties of linear-fractional transformations:
   (a) If $L(z) = L(w)$, then $(az + b)(cw + d) = (aw + b)(cz + d)$, and this simplifies to $(ad - bc)(z - w) = 0$. Because $ad - bc$ is not zero, $z - w$ must be. By the way, the inverse of $L$ is $L^{-1}(z) = \frac{dz - b}{-cz + a}$.
   (b) As part (a) shows, $L$ would otherwise not be one-to-one. In fact, $L$ would be a constant function, which makes it fairly uninteresting as a mapping.
   (d) Notice that $|e^{it} + b| = |e^{-it} + \overline{b}| = |e^{it}||e^{-it} + \overline{b}| = |1 + \overline{b}e^{it}|$. Therefore $L(e^{it})$ is on the unit circle, proving that $L$ maps the unit circle to itself. Notice also that $L^{-1}$ has the same form as $L$, which shows that the image of the unit circle is all of the unit circle. If $|b| < 1$, then the interior of the unit circle is mapped to the interior of the unit circle, for $F(0) = b$. Otherwise, $L$ maps the interior onto the exterior, and the exterior onto the interior. Notice that the case $|b| = 1$ is excluded by the other hypotheses.

3. That $L$ maps the unit disk to itself follows immediately from part (d) of item 2, given that $|q| < 1$. 

Dynamic Fractal Solutions
Figures
Where are the figures?

The figures for the following (mostly blank) pages can be obtained by using Winfeed. To view or print Figure 21, for example, open the file named “frfig21”.

Each file has been saved with specifications that automatically place it in the correct position on its page when you “File|Print” the figure. The program will ask whether it should “save while printing.” If you are printing on a 600-dpi printer, the printer file (which has the same name) has already been created, which allows you to say “no”. You can in fact use the menu item “File|Disk Image|Reprint” to directly print this file, which is somewhat quicker than re-creating it. Most pictures will take several minutes either way.
Figure 4.
Figure 6.
Figure 8.
Figure 9.
Figure 10.
Figure 12.
Figure 16.
Figure 21.
Figure 22.
Figure 24.
Figure 25.
Figure 26.
Figure 31.
Figure 33.
Figure 35.
Figure 40.
Figure 42.
Figure 45.
Figure 49.
Figure 50.
Figure 51.
Figure 52.
Figure 56.
Figure 58.
Figure 62.
Figure 63.
Figure 66.
Figure 67.
Figure 69.
Figure 70.
Figure 71.
Figure 72.
Figure 76.
Figure 78.
Figure 79.
Figure 80.
Figure 81.
Figure 82.
Figure 83.
\[
\frac{6}{7} \quad \frac{3}{4} \quad \frac{5}{7} \quad \frac{1}{2} \quad \frac{3}{7}
\]

5/14

\[
\frac{1}{4}
\]

3/16

\[
\frac{1}{8}
\]

Figure 86.