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Commensurable Triangles

Richard Parris

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The most famous equation in geometry is $a^2 + b^2 = c^2$, from the Pythagorean theorem. Three positive integers (a, b, c) that satisfy this equation are called a Pythagorean triple, and there are known methods to generate them all. For example, $(n^2 - m^2, 2mn, n^2 + m^2)$ is a Pythagorean triple whenever m and n are relatively prime positive integers, with one odd, one even, and $m < n$. Moreover, every Pythagorean triple whose greatest common divisor is 1 is of this form.

In this article, we consider another family of triples, representing triangles in which one angle is a rational multiple of another. These triples also satisfy defining polynomial equations, and our principal objective is to derive formulas that generate all primitive solutions to those equations.

Given a positive integer k, a triangle in which one angle is k times as large as another is called $(1, k)$-*commensurable*, or simply k-*commensurable*. This terminology is also applicable to triples. A triple is called *integral* or *rational* if its members are all integers or all rational numbers, and an integral triple is called *primitive* if there is no integer greater than 1 that divides all three members.

2-Commensurable triangles

Our first example is the 4–6–5 triangle ΔABC shown in Figure 1. It is 2-commensurable, since $\angle B$ is twice as large as $\angle A$. Notice also that $6^2 = 4(4 + 5)$. In fact, the equation

$$b^2 = a(a + c)$$

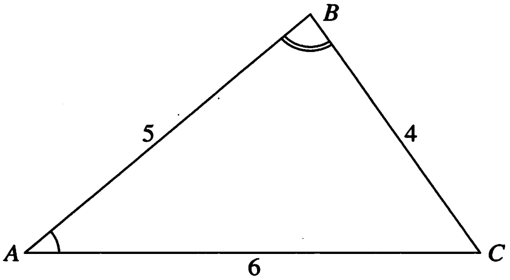

Figure 1.

is a necessary and sufficient condition for $\angle B$ to be twice $\angle A$ in ΔABC. (We are adopting the standard convention that a, b, and c are the lengths of the sides opposite angles A, B, and C.) One way to see this is to notice that the bisector of $\angle B$ splits the triangle into two triangles, one similar to the original, the other isosceles. The reader may enjoy completing this derivation.

A familiar irrational solution to this equation is $(a, b, c) = (1, \sqrt{2}, 1)$, for which $A = 45°$ and $B = 90°$. Another is $(a, b, c) = (1, \sqrt{3}, 2)$, for which $A = 30°$ and $B = 60°$.

Given that $b^2 = a(a + c)$ characterizes 2-commensurable triangles, it is natural to ask for primitive solutions to this equation. Notice that a and $a + c$ must be relatively prime in such a solution. Because every prime divisor of b^2 appears an even number of times, it follows that a must be a perfect square, as must $a + c$.

Any triple of the form $(m^2, mn, n^2 - m^2)$ is a solution to $b^2 = a(a + c)$, and it corresponds to a 2-commensurable triangle if $m < n < 2m$. If m and n are relatively prime, this triple is primitive, and every primitive 2-commensurable triple can be obtained in this way. Table 1 shows the first few examples. The condition $n < 2m$ is needed so that (a, b, c) will satisfy the triangle inequality $c < a + b$. Notice that $\angle A$ is not necessarily the smallest angle in the triangle, nor is $\angle B$ necessarily the largest.

Table 1.

m	n	a	b	c
2	3	4	6	5
3	4	9	12	7
3	5	9	15	16
4	5	16	20	9
4	7	16	28	33
5	6	25	30	11
5	7	25	35	24
5	8	25	40	39
5	9	25	45	56

3-Commensurable triangles

Now consider the 8–10–3 triangle shown in Figure 2. It is 3-commensurable, since $\angle B$ is three times as large as $\angle A$. Notice that $3^2 \cdot 8 = (10 + 8)(10 - 8)^2$. In fact, the equation

$$c^2 a = (b + a)(b - a)^2 \qquad (*)$$

is necessary and sufficient for $\angle B$ to be three times $\angle A$ in ΔABC. One way to derive $(*)$ is draw the trisector of $\angle B$ that lies closer to C. This splits the triangle into two smaller triangles, one similar to ΔABC, the other 2-commensurable. As in the 2-commensurable case, the reader is invited to provide the details.

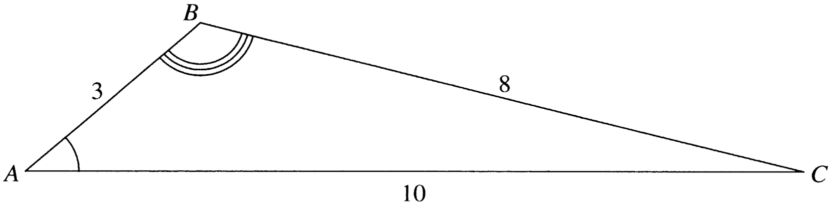

Figure 2.

A familiar irrational solution to $(*)$ is $(a, b, c) = (1, 2, \sqrt{3})$, for which $A = 30°$ and $B = 90°$. Another is $(a, b, c) = (2, 1 + \sqrt{5}, 2)$, for which $A = 36°$ and $B = 108°$.

Let (a, b, c) be a primitive solution to $(*)$, and let m be the greatest common divisor of a and b. Substituting $a = a'm$ and $b = b'm$ into $(*)$ leads to

$$c^2 a' = (b' + a')(b' - a')^2 m^2.$$

Because the triple (a, b, c) is primitive, c is relatively prime to m, and because a' and b' are relatively prime, the same must be true of a' and $(b' + a')(b' - a')^2$. It follows that

$$a' = m^2 \quad \text{and} \quad c^2 = (b' + m^2)(b' - m^2)^2.$$

This in turn implies that $b' + m^2$ must be a perfect square n^2. Thus a primitive 3-commensurable triple must take the form $(a, b, c) = \big(m^3, mn^2 - m^3, n^3 - 2m^2 n\big)$, where m and n are relatively prime. In order that $c > 0$ and the triangle inequality be satisfied, n must lie between $\sqrt{2}m$ and $2m$.

Table 2 shows the first few primitive 3-commensurable triples. As in Table 1, $\angle A$ need not be the smallest angle in the triangle, nor $\angle B$ the largest. Notice, however, that the pairs (m, n) in Table 1 for which $\angle A$ *is* the smallest angle also appear in Table 2. This persistence is a consequence of a recursive approach, which we describe below.

Table 2.

m	n	a	b	c
2	3	8	10	3
3	5	27	48	35
4	7	64	132	119
5	8	125	195	112
5	9	125	280	279
6	11	216	510	539
7	10	343	357	20
7	11	343	504	253
7	12	343	665	552
7	13	343	840	923

k-Commensurable triangles

Figure 3 shows a k-commensurable triangle ΔABC, in which $\overline{BD}$ is drawn so that $\angle DBC$ is congruent to $\angle A$. Triangles BDC and ABC are similar, with $BD/AB = a/b$. Thus $BD = ac/b$ and $DC = a^2/b$. Triangle ABD is $(k-1)$-commensurable. In particular, ΔABD is isosceles when $k = 2$. Hence the equation

$$b^2 - a^2 = ac$$

characterizes 2-commensurable triangles. This equation can in turn be applied recursively to the case $k = 3$, yielding

$$\left(b - \frac{a^2}{b}\right)^2 - \left(\frac{ac}{b}\right)^2 = c\left(\frac{ac}{b}\right).$$

Rearranging this equation and removing an extraneous factor $a + b$ leads to the promised 3-commensurable equation $(*)$.

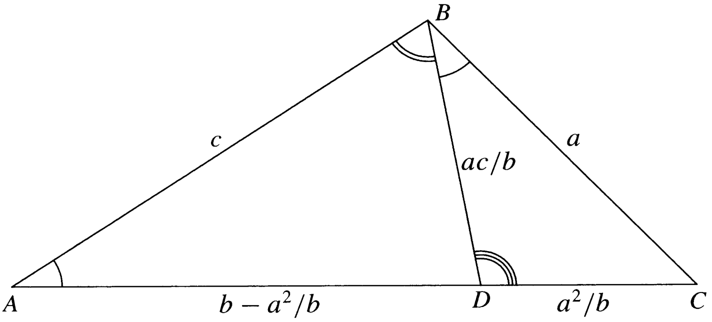

Figure 3.

Although any k-commensurable triangle can be derived from a $(k-1)$-commensurable triangle, not every $(k-1)$-commensurable ΔABC can be enlarged to a k-commensurable triangle. Such an enlargement of $\angle B$ is possible if, and only if, $\angle A$ is smaller than $\angle C$. As Figure 4 shows, this condition is needed so that the new ray through B will intersect $\overrightarrow{AC}$ at D.

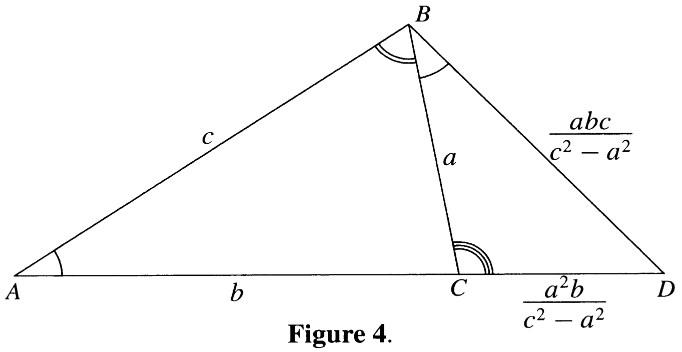

Figure 4.

Notice also that the labels on $\overline{CD}$ and $\overline{BD}$ in Figure 4 make sense only when $a < c$. The reader can obtain them by solving the proportion

$$\frac{CD}{BD} = \frac{BD}{b + CD} = \frac{a}{c},$$

derived from the similar triangles ΔBCD and ΔABD.

Recursive calculations

The preceding discussion shows that every rational k-commensurable triangle can be derived by enlarging a rational $(k-1)$-commensurable triangle. Applied to primitive $(k-1)$-commensurable triples (a, b, c) in which $a < c$, the formula

$$\left(\frac{abc}{c^2 - a^2}, \frac{bc^2}{c^2 - a^2}, c\right)$$

thus produces a rational multiple of every primitive k-commensurable triple. It follows that every primitive k-commensurable triple can be obtained by applying the following formula, removing common factors when necessary.

Recursion 0. If (a, b, c) is an integral $(k-1)$-commensurable triple, and if $a < c$, then $(ab, bc, c^2 - a^2)$ is an integral k-commensurable triple.

Let m and n be relatively prime. Then primitive 1-commensurable (isosceles) triples are described by $(a, b, c) = (m, m, n)$ for $n < 2m$, and primitive 2-commensurable triples are described by $(a, b, c) = (m^2, mn, n^2 - m^2)$ for $m < n < 2m$. When Recursion 0 is applied to $(a, b, c) = (m^2, mn, n^2 - m^2)$ however, the result is

$$(a, b, c) = \left(m^3 n, mn^3 - m^3 n, n^4 - 2m^2 n^2\right),$$

which is n times the earlier formula for primitive 3-commensurable triples.

The reader may wish to use this approach to verify some of the entries in Table 3. Notice that the kth row is a triple (a_k, b_k, c_k) of kth-degree polynomials in m and n.

Table 3.

a_k	b_k	c_k
m	m	n
m^2	mn	$n^2 - m^2$
m^3	$mn^2 - m^3$	$n^3 - 2m^2 n$
m^4	$mn^3 - 2m^3 n$	$n^4 - 3m^2 n^2 + m^4$
m^5	$mn^4 - 3m^3 n^2 + m^5$	$n^5 - 4m^2 n^3 + 3m^4 n$
m^6	$mn^5 - 4m^3 n^3 + 3m^5 n$	$n^6 - 5m^2 n^4 + 6m^4 n^2 - m^6$

Computing the first few rows of Table 3 led us to conjecture that every removed common polynomial factor would itself be found in the table. Given the initial data

$c_0 = 1$ and $(a_1, b_1, c_1) = (m, m, n)$, the rows of Table 3 appear to obey the following recursion:

Recursion 1. For $k > 1$, let $a_k = \dfrac{a_{k-1}b_{k-1}}{c_{k-2}}$, $b_k = \dfrac{b_{k-1}c_{k-1}}{c_{k-2}}$, and $c_k = \dfrac{c_{k-1}^2 - a_{k-1}^2}{c_{k-2}}$.

Working exclusively with this recursion proved to be awkward, however. A closer look at the table prompted an even simpler description:

Recursion 2. For $k > 1$, let $a_k = ma_{k-1}$, $b_k = mc_{k-1}$, and $c_k = nc_{k-1} - m^2c_{k-2}$.

The familiar look of the coefficients in this table eventually inspired this explicit formulation:

Recursion Lemma. *Recursions 1 and 2 produce the same polynomials, namely*

$$a_k = m^k,$$

$$c_k = \sum_{i=0}^{\lfloor k/2 \rfloor} (-1)^i \binom{k-i}{i} m^{2i} n^{k-2i},$$

and

$$b_k = mc_{k-1}.$$

Proof. An inductive demonstration is anchored for small k by the Table 3 data. The following four exercises complete the argument. ■

Exercise 1. By combining two successive applications of Recursion 2, eliminate n and obtain $c_{k-2}c_k + m^2c_{k-2}^2 = c_{k-1}^2 + m^2c_{k-3}c_{k-1}$. This is valid for $k \geq 3$.

Exercise 2. Obtain $c_{k-2}c_k = c_{k-1}^2 - m^{2k-2}$, the critical part of Recursion 1, by creating a telescoping sum from the result of Exercise 1. This is valid for $k \geq 2$.

Exercise 3. Assume for the purposes of induction that the triples (a_i, b_i, c_i) generated by the two recursions agree for all $i < k$. Combine Exercise 2 and Recursion 1 to show that the two recursions then agree on (a_k, b_k, c_k).

Exercise 4. Apply $c_k = nc_{k-1} - m^2c_{k-2}$ (from Recursion 2) to establish the explicit formula for c_k, assuming its validity for c_{k-1} and c_{k-2}.

The first theorem

Now consider a k-commensurable triangle obtained by evaluating the polynomials a_k, b_k, and c_k at a suitable pair (m, n) of relatively prime positive integers. The explicit formulas make it clear that the integers $a_k(m, n)$ and $c_k(m, n)$ are relatively prime. Hence the triple $(a_k(m, n), b_k(m, n), c_k(m, n))$ is primitive. The meaning of "suitable" is clarified next.

Cosine Lemma. *If ΔABC is a rational k-commensurable triangle, then* $(2\cos A)m = n$ *for some pair of relatively prime positive integers m and n.*

Proof. This is obvious when $k = 1$, and the reader might enjoy using the law of cosines to verify the formula in the case $k = 2$. However, because every k-commensurable triangle can be obtained by enlarging a $(k-1)$-commensurable triangle that shares $\angle A$, no additional work is actually needed. ■

This accomplishes one of the major objectives of this article:

Theorem 1. *For any positive integer k, the polynomials*

$$\begin{aligned} a_k &= m^k \\ b_k &= mc_{k-1} \\ c_k &= \sum_{i=0}^{\lfloor k/2 \rfloor} (-1)^i \binom{k-i}{i} m^{2i} n^{k-2i} \end{aligned}$$

generate all primitive k-commensurable triples, when they are evaluated at pairs (m, n) *of relatively prime positive integers that satisfy* $\left(\cos \frac{180^\circ}{k+1}\right) m < n < 2m$.

Proof. The sum of the angles of any triangle is 180°, and a k-commensurable triangle ΔABC contains $k+1$ copies of $\angle A$. Thus $(k+1)A < 180^\circ$ and $\cos \frac{180^\circ}{k+1} < \cos A < 1$. The result now follows from the Cosine Lemma. ■

The commensurable triangle theorem

If k/h is a rational number in lowest terms, then a triangle in which one angle is k/h times as large as another is called (h, k)*-commensurable*. The preceding discussion has dealt thoroughly with the case $h = 1$. Our goal is to use Table 3 to build a complete catalogue of primitive (h, k)-commensurable triples. The next example illustrates the case $h = 2$ of an impending theorem.

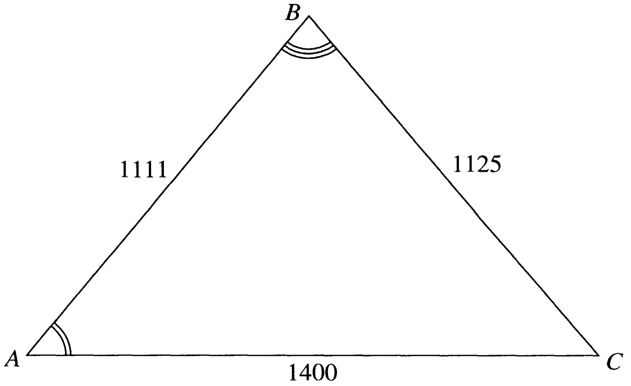

Figure 5.

The 1125–1400–1111 triangle shown in Figure 5 is $(2, 3)$-commensurable, since $\angle B$ is $1\frac{1}{2}$ times as large as $\angle A$. The reader can also verify that

$$(1400^2 - 1125^2)^2 + 1400 \cdot 1111(1400^2 - 1125^2) = 1125^2 \cdot 1111^2.$$

In fact, the equation

$$\left(b^2 - a^2\right)^2 + bc(b^2 - a^2) = a^2c^2$$

is necessary and sufficient for $\angle B$ to be $1\frac{1}{2}$ times $\angle A$ in ΔABC.

One way to see this is to notice that the trisector of $\angle B$ that is closer to A splits the triangle into two triangles, one similar to the original, the other 2-commensurable. This approach can also be used to show that primitive solutions to this equation have the form $(m^3n, m^2n^2 - m^4, n^4 - 3m^2n^2 + m^4)$, where m and n are relatively prime and $(2\cos 36°)m < n < 2m$. Table 4 displays a few examples.

Table 4.

m	n	a	b	c
3	5	135	144	31
4	7	448	528	305
5	9	1125	1400	1111
6	11	2376	3060	2869
7	12	4116	4655	1969
7	13	4459	5880	6119
8	13	6656	6720	209
8	15	7680	10304	11521
9	16	11664	14175	9889

The formula for primitive (2, 3)-commensurable triples has the form

$$(c_1(m,n)a_3(m,n), a_1(m,n)b_3(m,n), c_4(m,n)),$$

which is a special case of the following definition. For any relatively prime pair (h, k) of positive integers, let

$$\begin{aligned} a_{h,k}(m,n) &= c_{h-1}(m,n)a_k(m,n), \\ b_{h,k}(m,n) &= a_{h-1}(m,n)b_k(m,n), \quad \text{and} \\ c_{h,k}(m,n) &= c_{k+h-1}(m,n). \end{aligned}$$

(Notice that $(a_{1,k}, b_{1,k}, c_{1,k}) = (a_k, b_k, c_k)$.)

Theorem 2. *Given a pair (m, n) of relatively prime positive integers that satisfy the constraint $\left(2\cos\frac{180°}{k+h}\right)m < n < 2m$, the triple $(a_{h,k}(m,n), b_{h,k}(m,n), c_{h,k}(m,n))$ is primitive and (h, k)-commensurable. Conversely, any primitive (h, k)-commensurable triple can be obtained in this way.*

This is the principal result of this article. To avoid repetitious detail, the proof is outlined as a sequence of lemmas and exercises. Our familiar dissection tactic is used recursively.

Any rational k-commensurable triangle ΔABC has the property that $(2\cos A)m = n$ for some relatively prime positive integers m and n. These are still the only angles that can be used to build a rational (h, k)-commensurable triangle ΔABC.

Rational Lemma. *Let h and k be relatively prime positive integers, and ΔABC be a rational (h, k)-commensurable triangle. Then there are relatively prime positive integers m and n such that $A = h\theta$, $B = k\theta$, and $(2\cos\theta)m = n$.*

Exercise 5. Prove this lemma, assuming that $h < k$. Start by locating D on $\overline{AC}$ so that $\angle DBC$ is congruent to $\angle BAC$. Explain why ΔABD is $(h, k-h)$-commensurable and rational. To finish the proof, it will be necessary to know that h and k are relatively prime.

The proof of Theorem 2 now proceeds by dissecting ΔABC in a different fashion. Locate E on $\overline{BC}$ so that $\angle CAE = \theta$. Notice that ΔAEC is $(k+h-1)$-commensurable and ΔABE is $(h-1, k)$-commensurable. Because the shape of a triangle is determined once two of its angles are known, it follows that ΔAEC is rational, and that the lengths of its sides are proportional to $a_{k+h-1}(m, n)$, $b_{k+h-1}(m, n)$, and $c_{k+h-1}(m, n)$. In the same fashion, it follows by induction that the sides of ΔABE are proportional to $a_{h-1,k}(m, n)$, $b_{h-1,k}(m, n)$, and $c_{h-1,k}(m, n)$.

Exercise 6. The sides of a rational (h, k)-commensurable ΔABC are proportional to

$$\begin{aligned}
&a_{k+h-1}(m, n)b_{h-1,k}(m, n) + c_{k+h-1}(m, n)a_{h-1,k}(m, n),\\
&b_{k+h-1}(m, n)b_{h-1,k}(m, n),\\
&c_{k+h-1}(m, n)c_{h-1,k}(m, n),
\end{aligned}$$

for some relatively prime positive integers m and n. Show this by using a consistent labeling of the sides of ΔAEC and ΔABE.

A technical lemma is now needed to establish that the integer $c_{k+h-2}(m, n)$ is a common factor of the preceding triple of integers.

Lemma. *For integers $h > 1$ and $k \geq 1$,*

$$m^{2h-3}b_k = c_{h-1}c_{k+h-2} - c_{h-2}c_{k+h-1}.$$

Exercise 7. Prove the lemma by induction on h. Notice that the base case $h = 2$ is a direct consequence of Recursion 2.

Exercise 8. Use the lemma and induction to prove the polynomial identities

$$\begin{aligned}
a_{k+h-1}b_{h-1,k} + c_{k+h-1}a_{h-1,k} &= c_{k+h-2}a_{h,k},\\
b_{k+h-1}b_{h-1,k} &= c_{k+h-2}b_{h,k}, \quad \text{and}\\
c_{k+h-1}c_{h-1,k} &= c_{k+h-2}c_{h,k}.
\end{aligned}$$

Conclude that the sides of any integral (h, k)-commensurable triangle are proportional to the integers $a_{h,k}(m, n)$, $b_{h,k}(m, n)$, and $c_{h,k}(m, n)$ for suitable relatively prime integers m and n.

Exercise 9. Assuming that (h, k) and (m, n) are pairs of relatively prime integers, show that the triple $(a_{h,k}(m, n), b_{h,k}(m, n), c_{h,k}(m, n))$ is primitive.

This concludes the proof of Theorem 2.

Wrap-up, additional examples, and questions

One obvious question remains: Can a non-equilateral triangle have integral sides and three commensurable angles? The answer is *no*. For a short proof, see [**1**, p. 228].

Among the primitive 2-commensurable triples are those for which n has the extreme value $m + 1$. For any triple $(m^2, m(m+1), 2m+1)$, the law of cosines shows that

$$\cos A = \frac{m+1}{2m} \quad \text{and} \quad \cos B = \frac{-m^2 + 2m + 1}{2m^2}.$$

Thus, these triangles approach the degenerate triangle whose angles are $A = 60°$, $B = 120°$, and $C = 0°$.

At the other extreme, $n = 2m - 1$ produces triples $(m^2, 2m^2 - m, 3m^2 - 4m - 1)$. The corresponding triangles approach the shape of the degenerate triangle $(1, 2, 3)$, whose angles are $A = 0°$, $B = 0°$, and $C = 180°$.

The sequence of primitive 2-commensurable examples

$$(4, 6, 5),\ (25, 35, 24),\ (144, 204, 145),\ (841, 1189, 840),$$
$$(4900, 6930, 4901),\ \ldots$$

approaches the shape of an isosceles right triangle. The corresponding pairs (m, n) are $(2, 3), (5, 7), (12, 17), (29, 41), \ldots$, where (m, n) is followed by $(m + n, 2m + n)$. The ratios $\frac{n}{2m}$ approach $\frac{1}{2}\sqrt{2}$.

The sequence of primitive 3-commensurable examples

$$(8, 10, 3),\ (343, 665, 552),\ (17576, 35074, 30285),$$
$$(912673, 1825055, 1580208),\ \ldots$$

approaches the shape of a 30°–60° right triangle. The corresponding pairs (m, n) are $(2, 3), (7, 12), (26, 45), (97, 168), \ldots$, where (m, n) is followed by $(2m + n, 3m + 2n)$. The ratios $\frac{n}{2m}$ approach $\frac{1}{2}\sqrt{3}$.

For each k, there are infinitely many relatively prime pairs (m, n) for which (a_k, b_k, c_k) is a k-commensurable triangle, since there are infinitely many rational numbers between $\cos \frac{180°}{k+1}$ and 1. This interval becomes vanishingly small as k increases, forcing the table of k-commensurable examples to begin with large values of m. Methods of calculus show that $m > \left(\frac{k+1}{\pi}\right)^2$, so the smallest side of a k-commensurable triangle exceeds $\left(\frac{k+1}{\pi}\right)^{2k}$. For example, (43046721, 58429017, 16657264) is the smallest 8-commensurable triangle, produced by $m = 9$ and $n = 17$.

We conclude with three questions. We would like to know an answer for the third.

Question 1. Can the shape of every irrational commensurable triangle be approximated to arbitrary precision by a rational commensurable triangle?

Question 2. Given a triangle ΔABC, the lengths of whose sides form a primitive commensurable triple (a, b, c), is it possible that the area of ΔABC is rational?

Question 3. Given a pair (h, k) of relatively prime positive integers, it is now clear that there is a polynomial equation $P_{h,k}(a, b, c) = 0$ with integer coefficients that defines (h, k)-commensurable triangles ΔABC. For example, $P_{2,3}(a, b, c) = (b^2 - a^2)^2 + bc(b^2 - a^2) - a^2c^2$. This article provides an explicit description of all rational solutions to such equations. Is there an explicit formula for the polynomials $P_{h,k}(a, b, c)$ themselves?

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Newton's Method and the Golden Ratio

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Using Newton's Method to approximate square roots provides an attractive example of the interplay between algebra and geometry in calculus. The following serendipitous appearance of the golden ratio enhances that beauty.

Recall that for a positive real number a, the Newton sequence converging to $\sqrt{a}$ is generated by the iterates of the function

$$f(x) = \frac{1}{2}\left(x + \frac{a}{x}\right) \quad \text{for } x > 0.$$

The graph of this function is the branch of the hyperbola $x^2 - 2xy + a = 0$ in the first quadrant, which (independent of a) has the y-axis and the line $y = \frac{1}{2}x$ as its asymptotes (see figure). Curiosity about the axis ℓ of this hyperbola inspired our discovery.

Let θ be the angle between the x-axis and ℓ, and let α be the angle between the x-axis and the asymptote $y = \frac{1}{2}x$, as in the figure. Note that $\theta = \frac{\pi}{4} + \frac{1}{2}\alpha$, where $\alpha = \text{Arctan}\left(\frac{1}{2}\right)$. A little trigonometry shows that

$$\tan\theta = \frac{1 + \tan\frac{1}{2}\alpha}{1 - \tan\frac{1}{2}\alpha} \quad \text{and that} \quad \tan\frac{1}{2}\alpha = \sqrt{5} - 2.$$

With a little bit of algebra, this leads to our discovery that the slope of the hyperbola's axis is the golden ratio $\frac{1 + \sqrt{5}}{2}$!